Actuarial mathematics in social protection (Theory)

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0. INTRODUCTION

1. MATHEMATICS OF FINANCE

1.1 Interest rates (*)

We start by rethinking the following relations

(Future Value) = (Present Value) × (Interest Rate)

 $(Present Value) = (Future Value) \times (Discount Rate).$

Interest rate, such as the return to savings and the cost of borrowing, is the price at which resources are transferred between different points of time. In other words, financial assets have time values and the interest rate (or the discount rate) translates the values of an asset between the present and the future.

Let us consider the value of a financial asset, which is described as a function of time, P(t). Put $W(t_1, t_2) = \frac{P(t_2)}{P(t_1)}$. Then we can interpret $W(t_1, t_2)$ as the relative price that translates the value of this asset at one time t_1 into the equivalent value at another point of time t_2 . We attribute the following properties to W:

- (I-1) W(t,t) = 1 for $\forall t$;
- (I-2) $W(t_1, t_2) > 1$ for $t_1 < t_2$;

(I-3) $W(t_1, t_2) \cdot W(t_2, t_3) = W(t_1, t_3)$ for $\forall t_1, t_2, t_3$.

The first condition is a trivial statement. The second condition means that the asset is a scarce good. The last condition describes non-existence of arbitrage opportunity.

Note : From (I-1) and (I-2) we have $W(t_2, t_1) = W(t_1, t_2)^{-1}$ [Discount rate = reciprocal of Interest rate] These three conditions restrict the form of W. In fact, we have

Theorem.

$$W(t_1, t_2) = e^{\int_{t_1}^{t_2} \rho(t)dt}$$
 for some $\rho(t) > 0$.

Proof. Define the average rate of return R(t,T) by $W(t,T) = e^{R(t,T) \cdot (T-t)}$ or $R(t,T) = \frac{\log W(t,T)}{T-t}$. To get an instantaneous rate of return, we take the limit of $T \to t$. Hence, by (I-1),

$$\rho\left(t\right) = \lim_{T \to t} R\left(t, T\right) = \lim_{T \to t} \frac{\log W\left(t, T\right)}{T - t} = \frac{\partial}{\partial T} \log W\left(t, T\right) \mid_{T = t} = \frac{W_2\left(t, t\right)}{W(t, t)} = W_2\left(t, t\right),$$

where $W_2(t,T) = D_2 W(t,T) = \frac{\partial}{\partial T} W(t,T)$.

Now we write down the no arbitrage opportunity condition, $W(s,t) \cdot W(t,u) = W(s,u)$.

By differentiating both sides with respect to u and then putting u = t (or, operating $\frac{\partial}{\partial u} \cdot |_{u=t}$ on both sides), we have

$$W(s,t) \cdot W_{2}(t,u) = W_{2}(s,u)$$
$$W(s,t) \cdot W_{2}(t,t) = W_{2}(s,t)$$
$$\frac{\partial}{\partial t}W(s,t) = W(s,t) \cdot \rho(t)$$

By integrating, we have

$$W(s,t) = e^{\int_s^t \rho(\sigma) d\sigma}.$$

Further if we assume the condition:

(I-4) $W(t_1 + s, t_2 + s) = W(t_1, t_2)$ for $\forall t_1, t_2, s$, then we have $\rho(t) = \rho = const$. Hence $W(t, s) = e^{\rho(s-t)}$.

Note: In terms of P(t), the above equation can be written as: $\frac{d}{dt}P(t) = \rho(t) \cdot P(t)$. By integrating, we have $P(t) = P(0)e^{\int_0^t \rho(\sigma)d\sigma}$.

Assume that time is measured by year. We define the annual interest rate i_t for year t (i.e. [t, t+1]) by $1+i_t = e^{\int_t^{t+1} \rho(\sigma) d\sigma}$. In the case of constant interest we define the annual interest rate by $1+i = e^{\rho}$.

Remark 1.

An economic theory explains interest as measure of liquidity preference. By liquidity of a financial asset, we mean the ability of the asset to be readily turned into a sum of money. The nominal interest rate is what you give up by holding money instead of financial assets; it is the opportunity cost of holding money, or reward to temporally forgone liquidity. The degree of interest rate depends on the expectation of future value of the asset. In terms of individual preference, we have

 $\begin{array}{rcl} \rho: \mathrm{large} & \Longrightarrow & \mathrm{impatient} & \Longrightarrow & \mathrm{short\text{-sighted}} \\ \rho: \mathrm{small} & \Longrightarrow & \mathrm{patient} & \Longrightarrow & \mathrm{fore\text{-sighted}} \end{array}$

Remark 2.

Note that in the above we have made the perfect foresight assumption. We have assumed that the interest is predetermined and is actually paid with no default risk. In the context of social security pension, most of its asset is invested in safe investment (typically Treasury bonds), which are considered to satisfy these assumptions. Later we mention how to deal with more general cases where the interest is stochastic and there is a default risk. Furthermore, we have assumed no feedback of the financial operations of the fund on the determination of the interest rate. This may be controversial when we consider a social security fund retaining substantial reserves in the economy. In this course we do not look into such general equilibrium effects.

1.2 Compound interest

Suppose that you deposit one unit of money in a bank account that pays i percent interest per year. In the simple interest, the accumulated sum (capital plus interest) in n years is given by (1+ni). The interest is not transferred into the capital. In the compound interest, on the other hand, interest is transferred into the capital at the end of every year and interest paid in a year will also earn interest in the next year and so on. Hence, the accumulated sum in n years is given by $(1+i)^n$, which is greater than (1+ni).

Exc. 1.2.1. Check $(1+i)^n > (1+ni)$.

In practice, simple interest is used in short-term contracts (usually less than one year) and compound interest is used in long-term contracts.

Next we consider compounding m times within a year. Namely we divide one year into m equal periods and transfer the interest into the capital at the end of each period. If we denote by $i^{(m)}$ the annual rate of interest (before compounding), then the actual interest rate for a year is given by

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i = e^{\rho}.$$

Note that $i^{(1)} = i$ and $\lim_{n \to \infty} i^{(m)} = \rho$. In fact, $i = i^{(1)} > i^{(2)} > i^{(3)} > \cdots \longrightarrow \rho$.

Exc. 1.2.2 Verify numerically that $i = i^{(1)} > i^{(2)} > i^{(3)} > \cdots \longrightarrow \rho$.

Formula: $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Tip. To find out how many years it takes for the initial deposit to grow the double amount, there is a rule of thumb, called the rule of 69. If the interest is compounded continuously at the annual rate of $\rho\%$, you divide 69 by ρ . For example, if annual rate is 6% then it takes 69/6=11.5 years. You can replace 69 by 72, which is a bit overestimation (but on the safe side) but has more integral divisors. (Derive the rule of 69. Hint: log2=0.69).

1.3 Cash-flows (payment streams) and their present values

We assume that the interest rate is constant over time, unless otherwise specified. For interest rate i (or equivalently, the force of interest ρ), we define the discount factor by: $v = \frac{1}{1+i} = e^{-\rho} < 1$.

(1) Discrete-time case

(i) General

We begin by discrete-time case. A cash-flow is described by data specifying how much money is paid at what timing.

$$U = (U_k, t_k)_{k=0}^{\infty}$$
 where $0 \leq t_0 < t_1 < t_2 < \cdots$

These data specify the contract that the amount U_k should be paid at t_k . Note that U_k can be positive or negative. By convention, we understand that positive values represent cash in-flows and negative values represent cash out-flows. A diagrammatic illustration is convenient for understanding the cash-flow situation and perform its present value calculation.

The present value of the above cash-flow U is defined by

$$PV(U) = \sum_{k=0}^{\infty} U_k \left(\frac{1}{1+i}\right)^{t_k} = \sum_{k=0}^{\infty} U_k v^{t_k}$$

Exc.1.3.1. Construct a spreadsheet model for calculating the above present value.

Mathemetically, for a sequence $A = \{a_n\}_{n=0}^{\infty}$, the series defined by

$$G_A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is called its generating function (or generating power series). Indeed, each term of the sequence a_n can be recaptured from its generating function by

$$a_n = \frac{1}{n!} \frac{d^n G_A}{dx^n} \left(0 \right).$$

(ii) Annuities

An annuity is a series of constant amount of payments. Note that it is a very particular type of cash-flow in the sence that the amount of each cash-flow is constant ($U_k = const.$) and the payment takes place periodically ($t_{k+1} - t_k = const.$). Let us assume that $U_k = 1$ and $t_{k+1} - t_k = 1$.

The present value of the payment stream $t_k = k$ and $U_k = 1$ (for k = 0, 1, 2, ..., n-1) is called *n*-year annuity certain payable at the beginning of each year and denoted by

$$\ddot{a}_{\overline{n|}} = PV\left\{(1,k) \mid 0 \le k \le n-1\right\} = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v}.$$

When we want to stress that $\ddot{a}_{\overline{n|}}$ depends on *i*, we shall write $\ddot{a}_{\overline{n|}}(i)$.

An annuity payable for ever is called perpetuity and denoted by $\ddot{a}_{\overline{\infty}|} = \lim_{n \longrightarrow \infty} \ddot{a}_{\overline{n}|} = \frac{1}{1-v}$.

Exc. 1.3.2. Confirm the above formula by the above spreadsheet model.

Exc. 1.3.3. Tabulate $\ddot{a}_{n!}(i)$ for $0 \leq n \leq 30$ and $i = 1\%, 2\%, \dots, 10\%$.

In a similar way, we define various types of annuities as the present values of the following payment streams.

1°. ℓ -year deferred *n*-year annuity certain payable at the beginning of each year

Payment stream: $t_k = k + \ell$ and $U_k = 1$ (for $k = 0, 1, 2, \dots, n-1$)

Notation and formula:

$$_{\ell \mid} \ddot{a}_{\overline{n}|} = PV\left\{(1,k) \mid \ell \leq k \leq \ell + n - 1\right\} = \sum_{k=\ell}^{\ell+n-1} v^k = v^\ell \frac{1 - v^n}{1 - v}.$$

2°. *n*-year annuity certain payable *m*-times a year at the beginning of each period Payment stream: $t_k = \frac{k}{m}$ and $U_k = \frac{1}{m}$ (for k = 0, 1, 2, ..., mn - 1)

Notation and formula:

$$\ddot{a}_{\overline{n|}}^{(m)} = PV\left\{\left(\frac{1}{m}, \frac{k}{m}\right) \mid 0 \leq k \leq mn-1\right\} = \frac{1}{m}\sum_{k=0}^{mn-1} v^{\frac{k}{m}} = \frac{1}{m} \cdot \frac{1-v^n}{1-v^{\frac{1}{m}}}.$$

 3° . *n*-year annuity certain payable at the end of each year

Payment stream: $t_k = k + 1$ and $U_k = 1$ (for k = 0, 1, 2, ..., n - 1) Notation and formula:

$$a_{\overline{n|}} = PV\{(1,k) \mid 1 \le k \le n\} = \sum_{k=1}^{n} v^k = v \frac{1-v^n}{1-v}.$$

Remark. In the actuarial context, we encounter $\ddot{a}_{\overline{n|}}$ more often than $a_{\overline{n|}}$. (Give reasons why) Exc. 1.3.4. Show $\ddot{a}_{\overline{n|}} = a_{\overline{n-1|}} + 1 = v^{-1}a_{\overline{n|}}$ algebraically and diagrammatically. Explain. Exc. 1.3.4b Define and derive the formula for $\frac{\ell}{m} |\ddot{a}_{\overline{n|}}^{(m)}$.

(iii) Varying annuities

We consider two types of increasing annuity.

The first is the case in which the amount increases in arithmetic progression. Hence the payment stream $t_k = k$ and $U_k = k + 1$ (for k = 0, 1, 2, ..., n - 1). We denote the present value of this cash-flow by

$$(I\ddot{a})_{\overline{n|}} = PV\left\{(k,k-1) \mid 1 \leqq k \leqq n\right\} = \sum_{k=1}^{n} kv^{k-1} = \frac{1}{1-v}\left(\frac{1-v^{n}}{1-v} - nv^{n}\right).$$

Exc. 1.3.5. Show that $(I\ddot{a})_{\overline{n|}} = \frac{d}{dv}a_{\overline{n|}}$. Calculate $\frac{d}{di}\ddot{a}_{\overline{n|}}$. (Hint: Chain rules)

Exc. 1.3.6. Show that $(I\ddot{a})_{\overline{n|}} = \sum_{k=0}^{n-1} {}_{k|}\ddot{a}_{\overline{n-k|}}$ algebraically and diagrammatically.

The second case is where the amount increases in geometric progression. This is a typical case in which the amount is adjusted in line with a steadily increasing index (e.g. wage). The payment stream is $t_k = k$ and $U_k = (1 + \beta)^k = w^k$ (for k = 0, 1, 2, ..., n - 1). We denote (in only this note) its present value by

$$\ddot{a}_{\overline{n|}}^{[\beta]} = PV\left\{(w^k, k) \mid 0 \le k \le n-1\right\} = \sum_{k=0}^{n-1} w^k v^k = \frac{1 - (wv)^n}{1 - wv}.$$

By suitably adjusting the discount rate (using the net discount rate), this case is reduced to the case of normal annuity.

Exc. 1.3.7. Show that $\ddot{a}_{\overline{n|}}^{[\beta]}(i) = \ddot{a}_{\overline{n|}}(j)$ where $\frac{1}{1+j} = \frac{1+\beta}{1+i}$ or $j \approx i - \beta$. Verify numerically.

(2) Continuous-time case

In the continuous-time case, a cash-flow is described by a function f(t) of time $t \in (0, \infty]$. The cash payable between t and t + dt is given by f(t)dt. We define the present value of the continuous cash-flow f(t) by

$$PV\left(f\right) = \int_{0}^{\infty} f\left(t\right) e^{-\rho t} dt = (Lf)\left(\rho\right).$$

Mathematically, this operation is called the Laplace transform (see Mathematical notes)

The continuous analogue of *n*-year annuity certain is defined as the present value of the continuous cash-flow given by $\lambda_n(t) = \begin{cases} 1 & (if \quad t \leq n) \\ 0 & (if \quad t > n) \end{cases}$. We can write $\lambda_n(t) = H(n-t)$, where H(t) is the Heaviside function: $H(t) = \begin{cases} 1 & (if \quad t \geq 0) \\ 0 & (if \quad t < 0) \end{cases}$.

The n-year continuous annuity certain is calculated by

$$\overline{a_{n|}} = PV\left(\lambda_n\right) = \int_0^n e^{-\rho t} dt = \frac{1}{\rho} \left(1 - e^{-\rho n}\right).$$

Exc. 1.3.8. Show that $\overline{a_{n|}} = \lim_{m \to \infty} \ddot{a}_{n|}^{(m)} = \lim_{m \to \infty} a_{n|}^{(m)}$.

Remark(*). The discrete case can be included in the continuous case by using the delta function (see Mathematical notes). For the discrete cash-flow $U = (U_k, t_k)_{k=0}^{\infty}$, the continuous version is $f(t) = \sum_{k=0}^{\infty} U_k \,\delta(t - t_k)$. (Check)

(3) Generalisation to time dependent interest rartes

The above results will hold if the interest rate varies over time. We only need to make the following modifications to the notation.

$$PV(U) = \sum_{k=0}^{\infty} U_k V_k \quad where \quad V_k = \prod_{s=1}^{t_k} v_s$$
$$PV(f) = \int_0^{\infty} f(t) e^{-\int_0^t \rho(s) ds} dt.$$

Note that in the continuous case, we can no longer see it the Laplace transform. We will rather have to see it as a functional $(Lf) [\rho(t)]$, that is, a function of function $\rho(t)$.

1.4 Contingent payments and expected present values (*)

Now we modify the contract that the amount is payable on condition that certain events occur. To specify the payment stream the probability of the contingency should also be given or estimated.

In the case of discrete payment and binary choice (i.e. to pay the full amount or not to pay at all), the pair $U = (U_k, t_k, p_k)$ $0 \leq t_0 < t_1 < t_2 < \cdots$ describes the congingency cash-flow. To calculate its expected present value, we first calculate the stream of the expected payment. $E(U) = (U_k p_k, t_k)$ since $E(U_k) = U_k p_k + 0 \cdot (1 - p_k) = U_k p_k$. Then $(E(U_k), t_k)$ is a normal payment stream. Next, we calculate its present value

$$EPV(U) = PV(E(U)) = \sum_{k=0}^{\infty} U_k p_k v^k$$

Consider a set of random variables aligned over time (stochastic process) $\{X_t\}_{t \in [0,\infty)}$. Let $\phi_t(x)$ the p.d.f. of X_t . Then the expected value of is $f(t) = E[X_t] = \int_{-\infty}^{\infty} x \phi_t(x) dx$. Hence,

$$EPV(\{X_t\}) = PV(E[X_t]) = PV(f) = \int_0^\infty f(t)e^{-\rho t}dt$$

Alternatively, define $Z = PV(X_t) = \int_0^\infty X_t e^{-\rho t} dt$. This is a random variable on $(-\infty, \infty)$. Let $\varphi(z)$ be the p.d.f. of Z. Then

$$EPV(\{X_t\}) = E[PV(X_t)] = E[Z] = \int_{-\infty}^{\infty} z\varphi(z)dz$$

(see also section 3.7)

1.5 Recursion formulae - applications to individual savings account and loan amortization

In practice, we can write down present value formulae only for very special types of cash-flows. Recursion formula can be used to treat more general cases. Recursion formula is particularly suitable to construct the algorithms. We consider here two kinds of situations.

(1) Accumulation of savings account

Consider an individual savings account where the initial capital F_0 is invested and an additional amount of C_t is invested at the end of year t (for t = 1, ..., n). Let F_t be the balance at the end of year t including the payment C_t . The increase in the balance during the year t consists of the interest credited on the previous year's balance and the payment made that year. Thus

$$\Delta F_{t-1} = F_t - F_{t-1} = iF_{t-1} + C_t$$

or $F_t = (1+i)F_{t-1} + C_t$.

Dividing both sides of the first equation by $(1+i)^t$ and taking summation from t = 1 to n, we have

$$\frac{F_t}{(1+i)^t} - \frac{F_{t-1}}{(1+i)^{t-1}} = \frac{C_t}{(1+i)^t}$$
$$\frac{F_n}{(1+i)^n} - F_0 = \sum_{t=1}^n \frac{C_t}{(1+i)^t}$$
$$F_n = (1+i)^n F_0 + \sum_{t=1}^n (1+i)^{n-t} C_t$$

This means that the balance at the end of year n consists of the initial fund and cash-flow payments that accumulate the interest from the time they were invested to the time of evaluation.

Alternatively, by summing the first equation, we have

$$F_t - F_0 = \sum_{t=1}^n iF_{t-1} + \sum_{t=1}^n C_t$$

The increase between the final and initial balance consists of the sum of cash-flow and the sum of each year's interest on the balance at the end of the corresponding year.

Exc. 1.5.1. Consider the balance of an individual savings. Let B_t be the balance at the end of year t (for t = 1, ..., n) and the initial capital $B_0 = 0$. It is more realistic to assume that the contribution of C_t is paid at the middle of year t and the rate of return varies over time i_t . In this situation, discuss the recuision formula that describes the evolution of the balance from one year to the next. (Note. in this case one should take into account half-a-year's interest on the contributions). Derive the formula for B_t in terms of C_t and i_t .

Exc. 1.5.2. Construct a spreadsheet model that calculates the balance of the individual retirement savings account for a model worker from age 20 to 60.

Consider payments are made and interest is credited continuously. Cash-inflow to the fund and interest credited to the fund between t and t + dt is given C(t)dt and $\rho F(t)dt$ respectively.

$$dF(t) = \rho F(t)dt + C(t)dt$$
$$\frac{dF(t)}{dt} = \rho F(t) + C(t)$$

Solving this differential equation with the initial condition at t = 0, we have

$$F(t) = e^{\rho t} F(0) + \int_0^t e^{\rho(t-s)} C(s) ds.$$

Exc. 1.5.3. Interpret this result in analogy with the discrete case.

Exc. 1.5.4. Consider the case where interest is time-dependent. Ans. $F(t) = e^{\int_0^t \rho(s)ds}F(0) + \int_0^t e^{\int_s^t \rho(u)du}C(s)ds$.

(2) Loan amortization

Suppose that you borrow P_0 and repay (or amortize) this debt (principal) by means of installment payment $R_1, R_2..., R_n$ to be made at the end of each year. Let P_t (t = 1, 2, ..., n) be the outstanding principal i.e., remaining debt immediately after the payment R_t has been made. In order to complete the repayment by the end of year n, it must be $P_n = 0$.

Since P_t consists of the previous year's debt with interest accumulated for one year less R_t , we have

$$P_t = (1+i)P_{t-1} - R_t.$$

By rearranging

$$\Delta P_{t-1} = P_t - P_{t-1} = iP_{t-1} - R_t$$

This implies that the change in the outstanding principal (primary balance) is accounted for the increase due to the interest on running debt and the decrease due to the repayment. So, in order not to increase the debt, the borrower has to pay at least the interest on the running debt.

Similarly, multiplying the both sides of the first equation by v^t (and noting that $v = \frac{1}{1+i}$) and suming from t = 1 to n, we have

$$v^{t-1}P_{t-1} - v^t P_t = v^t R_t$$

 $P_0 - v^n P_n = \sum_{t=1}^n v^t R_t.$

Using $P_n = 0$, we have the following present value condition.

$$P_0 = \sum_{t=1}^n v^t R_t = PV(\{R_t\}_{t=1}^n).$$

The repayments R_t may be chosen arbitrary subject to the present value constraint. We list some standard types of loans.

(i) Under the fixed loan contract, only the interest on the initial debt is paid until the principal is repaid at the end of the contract.

$$R_1 = R_2 = \dots = R_{n-1} = iP_0$$
; $R_n = (1+i)P_0$
 $P_1 = P_2 = \dots = P_{n-1} = P_0$; $P_n = 0$

(ii) Under an annuity loan contract, the principal is repaid by constant payments. From the present value constraint, the amount of repayment is therefore,

$$R_1 = R_2 = \dots = R_{n-1} = R_n = \frac{P_0}{a_{\overline{n}|}}.$$

The debt outstanding decreases exponentially by

$$P_t = P_0(1+i)^t \left(1 - \frac{a_{\overline{t}|}}{a_{\overline{n}|}}\right) \qquad (1 \le t \le n).$$

Exc. 1.5.5. Check the above result.

(iii) A series loan is a contract in which the principal outstanding decreases linearly

$$P_t = P_0(1 - \frac{t}{n}) \qquad (1 \le t \le n).$$

The repayment is thus

$$R_t = iP_t - \frac{1}{n}P_0 = P_0 \cdot \frac{i(n-t+1) - 1}{n} \qquad (1 \le t \le n).$$

Remark 1. Observe that in the limit $n \to \infty$, all the three contracts degenerates to an infinite loan without repayment. Amortization consists only of interest which is paid indefinitely.

Remark 2. From the result of the annuity loan contract, we see that if we annuitise the capital P_0 over n years then the annual amount is $\frac{P_0}{a_{\overline{n}|}}$. Conversely, if one pays annuity of one unit currency at the end of year for n times, then $P_0 = a_{\overline{n}|} (< n)$ is the initial capital necessary for this payment. (Reformulate this for the continuous case)

1.6 Internal rate of return

So far we have investigated in methods of evaluating the present value under a given interest rate. Now we consider the inverse problem: find the interest rate that equates the present value of a cash-flow with its price.

(1) Internal rate of return of a financial asset

Suppose a financial asset which entitles n future payments, denoted by $s_1, ..., s_n$. Assume the payment s_k is due at time t_k (for k = 1, 2, ..., n). If one purchases this asset at a price p, then the rate of return can be calculated in the following methods.

(i) Simple interest (payments bear no interest)

The condition is

$$(1+it_n) \cdot p = \sum_{k=1}^n s_k$$
; hence, $i = \frac{\sum_{k=1}^n s_k - p}{t_n \cdot p}$.

(ii) Simple interest (payments bear simple interest)

The condition is

$$(1+it_n) \cdot p = \sum_{k=1}^n s_k \cdot (1+i(t_n-t_k)) \quad ; \quad hence, \quad i = \frac{\sum_{k=1}^n s_k - p}{t_n \cdot p - \sum_{k=1}^{n-1} s_k(t_n-t_k)}.$$

(iii) Compoud interest (payments bear compound interest)

The condition is

$$(1+i)^{t_n} \cdot p = \sum_{k=1}^n s_k \cdot (1+i)^{t_n-t_k}$$

or, $p = \sum_{k=1}^n s_k v^{t_k} = \phi(v).$

The solution to the above non-linear equation can be found by standard numerical methods. Excel has a built-in function, IRR(), which calculates internal rates of return.

(a) Line search (Regula falsi)

Note that $\phi(v)$ is an increasing function of v ($0 \le v \le 1$) and that $\phi(0) = 0 < p$ and usually $\phi(1) = \sum s_k > p$. Depending on whether $\phi(\frac{1}{2}) > p$ or < p, the range of the solution to this equation is $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. Continuing this we can limit the range of the solution.

(b) The Newton-Raphson method

The solution is approximated by the following iteration scheme:

$$v_{i+1} = v_i - \frac{\phi(v_i) - p}{\phi'(v_i)} = v_i - \frac{\sum_{k=1}^n s_k v^{t_k} - p}{\sum_{k=1}^n s_k t_k v^{t_k - 1}}$$

The initial value may be taken as $v_0 = 1$.

(c) A method using the convexity of $\phi(v)$

A simpler iteration scheme using the fact that $\phi(v)$ is a convex increasing function of v $(0 \leq v \leq 1)$ is given as follows:

$$v_{i+1} = \frac{\phi(v_{sol})}{\phi(v_i)} v_i = \frac{p}{\sum_{k=1}^n s_k v_i^{t_k}} v_i$$

where the initial value is $v_0 = 1$.

If the holder sells this asset at price q at time τ ($t_{m-1} < \tau \leq t_m$), then the resulting rate of return is calculated by applying the same method for the cash stream $S_k = s_k$ due at t_k (for k = 1, 2, ..., m-1) and $S_m = q$ due at τ .

Example (bonds)

When a coorporation or a government needs to raise money, they issue bonds and sell them to a large number of investors. A bond is a certificate, in which the borrower agrees to pay specified interest (called coupon rate) until a specified date (called maturity date) when a fixed amount is paid (redemption value). Conventionally the interest rate is applied to the face value which is printed on the bonds. Usually the redemption value is equal to the face value but it is not necessarily the case.

Suppose an investor pays P for the purchase of a security bond with a face amount F, yearly coupons C (payable at the end of the year) and remaining running time n years. In the above notation, the cash stream is written $t_k = k$ (for k = 1, 2, ..., n); $s_1 = s_2 = \cdots = s_{n-1} = C$ and $s_n = C + F$.

The formulae corresponding to the cases (i) and (ii) can be written as follows:

Case (i)

$$i = \frac{nC + F - P}{nP} = \frac{C + \frac{F - P}{n}}{P}$$

Case (ii)

$$i = \frac{nC + F - P}{nP - \frac{n(n-1)}{2}C} = \frac{C + \frac{F - P}{n}}{P - \frac{n-1}{2}C}$$

Case (iii) (iteration scheme using method (c))

$$v_{i+1} = \frac{P}{\sum_{k=1}^{n} Cv_i^k + Fv_i^n} \cdot v_i$$

Exc. 1.6.1. Evaluate rates of return of various bonds under methods (i) and (iii).

(2) Rate of return of a financial institution

Suppose a financial institution which deals with cash-flows and retains certain assets as reserves. Consider an operation of a year. Time for that year is denoted by $0 \le t \le 1$.

Let

- F(t): reserve at time t
- C(t): (instantaneous) net cash-inflow to the fund incurring at time t

C:cash-flow during the year $\left(=\int_{0}^{1}C(t)dt\right)$

I :investment income during the year

A : reserve at the beginning of the year (=F(0))

B : reserve at the end of the year (= F(1)).

If there is no other financial flow to the institution, by accounting requirement, B - A = C + I holds as an identity.

The force of return, denoted by ρ , is defined by

$$dF(t) = \rho F(t)dt + C(t)dt.$$

Hence,

$$F(t) = F(0)e^{\rho t} + \int_0^t C(s)e^{\rho(t-s)}ds$$

If A, B, C(t) are known, the value of ρ is determined by

$$B = Ae^{\rho} + \int_0^1 C(t)e^{\rho(1-t)}dt,$$

and the investment income is given by

$$I = B - A - C = \rho \int_0^1 e^{\rho(1-t)} [A + \int_0^t C(s)ds] dt$$

To find ρ in the above equation, we apply the following iteration scheme (the Newton-Raphson method):

$$\rho_{i+1} = \rho_i - \frac{Ae^{\rho_i} + \int_0^t C(t)e^{\rho_i(1-t)}dt - B}{Ae^{\rho_i} + \int_0^t C(t)(1-t)e^{\rho_i(1-t)}dt}$$

A first order approximation, valid if ρ is small, will result

$$I = B - A - C$$

= $\rho \int_0^1 e^{\rho(1-t)} [A + \int_0^t C(s) ds] dt \equiv \rho W$

Therefore,

$$i = e^{\rho} - 1 \simeq \rho = \frac{I}{W}.$$

Note that $W = \int_0^1 e^{\rho(1-t)} [A + \int_0^t C(s) ds] dt$ means the term-weighted average of the reserves in that year.

Example

Suppose that the cash-flow occurs at the middle of the year, i.e. $C(t) = C \cdot \delta(t - \frac{1}{2})$, where $\delta(t)$ is the delta function. Then,

$$B = Ae^{\rho} + Ce^{\frac{1}{2}\rho} = A(1+i) + C\sqrt{1+i}$$

By solving the above equation with respect to i (assuming linear approximation of the square root, $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$, we have

$$i = \frac{2I}{A+B-I}.$$

This is so-called "Hardy's formula".

Exc. 1.6.2. Evaluate rates of return for social security pension schemes.

2. SURVIVAL MODELS AND LIFE TABLES

2.1 The survival function

Survival function represents the boundary that divides the likelihood of a person's survival status into alive and dead.

Let $R_{\geq 0} = [0, \infty)$, and consider the function $\ell : R_{\geq 0} \longrightarrow R_{\geq 0}$ (i.e., $\ell(x) \geq 0$ for $x \geq 0$), which has the following properties.

(S-1) Strictly decreasing ; if x < y then $\ell(x) > \ell(y)$.

(S-2) $\lim_{x\to\infty} \ell(x) = 0$. If there exists $\omega > 0$ such that $\ell(x) = 0$ for $x \ge \omega$, then ω is called the limiting age.

(S-3) $\ell(x)$ is continuously differentiable (i.e., C^1 -class).

Note: From (i) and (iii), it follows that $\ell'(x) < 0$.

For the moment we assume the existance of such functions. We may often write ℓ_x instead of $\ell(x)$.

For an age x > 0 (Note: x does not need to be an integer) and $t \ge 0$, define

$${}_t p_x = \frac{\ell_{x+t}}{\ell_x},$$

$${}_t q_x = 1 - {}_t p_x = 1 - \frac{\ell_{x+t}}{\ell_x}$$

These quantities have the following meaning:

 $_{t}p_{x}$ = probability that a life aged x will attain age x + t $_{t}q_{x}$ = probanility that a life aged x will die within t years In a special case of t = 1, we denote

$$q_x = {}_1q_x \quad ; \qquad p_x = {}_1p_x$$

Clearly, for t, u > 0,

$$_{t+u}p_x = \quad _up_{x+t} \cdot _tp_{x}.$$

Exc. 2.1.1. Interpret the probablistic meaning of the above relation.

2.2 Force of mortality

The force of mortality at age x > 0 is defined by

$$\mu(x) = \mu_x = -\frac{1}{\ell_x} \frac{d\ell_x}{dx} = -\frac{d\log\ell_x}{dx}.$$

Firstly, from the definition, we have

$$-d\ell_x = \mu_x \ell_x dx.$$

Integrating both sides from x to x + t, we have $\ell_x - \ell_{x+t} = \int_0^t \mu_{x+s} \ell_{x+s} ds$. Hence,

$${}_{t}q_{x} = \frac{\ell_{x} - \ell_{x+t}}{\ell_{x}} = \int_{0}^{t} \mu_{x+s} \frac{\ell_{x+s}}{\ell_{x}} ds = \int_{0}^{t} \mu_{x+s} s p_{x} ds$$

and

$$_{t}p_{x} = 1 - _{t}q_{x} = \int_{t}^{\infty} \mu_{x+s} \, _{s}p_{x}ds.$$

It is worth noting that we frequently encounter the expression $\mu_{x+t} p_x$ in this combination. Later we will investigate the probalistic meaning of this quantity (see 2.5).

Secondly, we start from

$$d\log\ell_x = -\mu_x dx.$$

Similarly, by integrating both sides from x to x + t, we have

$$\log\left(\frac{\ell_{x+t}}{\ell_x}\right) = -\int_x^{x+t} \mu_s ds = -\int_0^t \mu_{x+s} ds,$$
$$\ell_{x+t} = \ell_x e^{-\int_0^t \mu_{x+s} ds}, \quad or$$
$$_t p_x = e^{-\int_0^t \mu_{x+s} ds}.$$

From this expression, we have the following important relations (Cf. differentiation under an integral).

$$\frac{\partial_t p_x}{\partial x} = \frac{\partial}{\partial x} \left(e^{-\int_x^{x+t} \mu_s \, ds} \right) = e^{-\int_x^{x+t} \mu_s \, ds} \left(-\mu_{x+t} + \mu_x \right) = {}_t p_x \left(\mu_x - \mu_{x+t} \right).$$
$$\frac{\partial_t p_x}{\partial t} = \frac{\partial}{\partial t} \left(e^{-\int_x^{x+t} \mu_s \, ds} \right) = e^{-\int_x^{x+t} \mu_s \, ds} \left(-\mu_{x+t} \right) = {}_t p_x \, \mu_{x+t}.$$

These (partial differential) equations are known as Kolmogorov-Chapman equations.

Exc.2.2.1. Show $\frac{d}{dx}(\ell_x \mu_x) = -\frac{d^2}{dx^2}\ell_x$; $\mu_{x+t} = -\frac{\partial \log_t p_x}{\partial t}$.

2.3 Life expectancy

The (complete) life expectancy at age $x \ge 0$ is defined by

$$e_x^{\circ} = \frac{1}{\ell_x} \int_0^\infty \ell_{x+t} dt = \int_0^\infty {}_t p_x dt.$$

Life expectancy expresses the average residual life at age x > 0. Geometrically, the life expectancy at age x is equivalent to the horizontal length of the rectangle that has the same area under the tail of the survival curve (figure).

Exc. 2.3.1. Show $e_x^{\circ} = \int_0^{\infty} t \,\mu_{x+t} t p_x dt$ (Hint. Integration by parts) ; $\frac{de_x^{\circ}}{dx} = \mu_x e_x^{\circ} - 1$ (Hint. Differentiate and use the relation in 2.2.)

Exc. 2.3.2. Show that e_x° is not necessarily a decreasing function of x, but $x + e_x^{\circ}$ is an increasing function of x. Give geometrical and intuitive explanation.

We call t the half-survival period at age x (or median future lifetime) if $\ell_x : \ell_{x+t} = 2 : 1$. Explicitly, we can write $t = \ell^{-1}(\frac{\ell_x}{2}) - x$.

2.4 Some analytical laws of mortality

(i) de Moivre

This law is given by $\ell_x = \ell_0 \left(1 - \frac{x}{\omega}\right)^{\alpha}, \quad (\alpha > 0).$

Exc. 2.4.1. Verify $\mu_x = \frac{\alpha}{\omega - x}$, $e_x^{\circ} = \frac{\omega - x}{\alpha + 1}$.

The historical case due to de Moivre (1724) is the linear case with $\alpha = 1$. Despite its analytical tractability, however, the demographic grounds of this model are weak.

(ii) Weibull

This law is defined by $\ell_x = \ell_0 \exp(-(\frac{x}{\rho})^{\theta})$ The force of mortality is then given by $\mu_x = \frac{\theta}{\rho} (\frac{x}{\rho})^{\theta-1}$

(iii) Gompertz-Mekeham

This law takes the form $\mu_x = A + Be^{\gamma x}$, $(A, B, \gamma > 0)$. Note that this can also be written $\mu_x = A + Bc^x$ ($\gamma = \log c$). By suitable choice of parameters, this model fits well for higher ages and thus is widely used in the context of life and pension insurance.

Range of parameters: $10^{-4} \lesssim A \lesssim 10^{-3}$, $10^{-6} \lesssim B \lesssim 10^{-5}$, $1.08 \lesssim c \lesssim 1.12$.

Let us calculate the survival function under this law.

$$\ell_x = \ell_0 \exp\left(-\int_0^x \mu_s ds\right)$$
$$= \ell_0 \exp\left(-\int_0^x (A + Be^{\gamma s}) ds\right)$$
$$= \ell_0 e^{-Ax - \frac{B}{\gamma}(e^{\gamma x} - 1)} = ks^x g^{c^x}.$$

Here, $c = e^{\gamma} > 1$, $s = e^{-A} < 1$, $g = e^{-\frac{B}{\gamma}} < 1$, $k = \ell_0 e^{\frac{B}{\gamma}}$.

From this result, we next derive the life expectancy.

$$e_x^{\circ} = \int_0^{\infty} {}_t p_x dt = \int_0^{\infty} \exp\left(-\int_0^t {}_{\mu_{x+s}} ds\right) dt = \int_0^{\infty} \exp\left(-\int_0^t {}_{\lambda} + B e^{\gamma(x+s)} ds\right) dt.$$

In the above

$$\int_0^t \left(A + Be^{\gamma(x+s)}\right) ds = At + Be^{\gamma x} \int_0^t e^{\gamma s} ds = At + \frac{Be^{\gamma x}}{\gamma} (e^{\gamma t} - 1) \equiv At + P(e^{\gamma t} - 1) \qquad (P = \frac{Be^{\gamma x}}{\gamma}).$$

Hence

$$e_x^{\circ} = \int_0^{\infty} \exp\left(P - At - Pe^{\gamma t}\right) dt = e^P \int_0^{\infty} e^{-At - Pe^{\gamma t}} dt$$

Changing variable by $z = Pe^{\gamma t}$, or $dz = \gamma Pe^{\gamma t}dt = \gamma zdt$, thus $dt = \frac{dz}{\gamma z}$, we have

$$\begin{split} e_x^{\circ} &= e^P \int_0^{\infty} e^{-At} e^{-P e^{\gamma t}} dt \\ &= e^P \int_P^{\infty} \left(\frac{z}{P}\right)^{-\frac{A}{\gamma}} e^{-z} \frac{dz}{\gamma z} \\ &= \frac{1}{\gamma} P^{\frac{A}{\gamma}} e^P \int_P^{\infty} z^{-\frac{A}{\gamma} - 1} e^{-z} dz = \frac{1}{\gamma} P^{\frac{A}{\gamma}} e^P \Gamma(-\frac{A}{\gamma}, P). \end{split}$$

Where $P = \frac{Be^{\gamma x}}{\gamma}$ and $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the (upper) incomplete gamma function.

Historical notes:

In deriving this law, Benjamin Gompertz (1825) considered that "the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals". If we regard $\frac{1}{\mu_x}$ as the methaphor of the 'power to avoid death' that he refers to, then the law is written down as $\frac{d}{dx}\left(\frac{1}{\mu_x}\right) = -\gamma \frac{1}{\mu_x}$. By solving this equation, we have $\mu_x = Be^{\gamma x}$.

In the same paper, he says that "it is possible that death may be the consequence of two generally coexisting causes: the one, chance, without previous disposition to death or deterioration; the other, a deterioration or increased inability to withstand destruction". Though the second factor is adequately taken into account in the above model, he did not consider the first factor. It was Mekeham (1860) who, along with this suggestion, modified the above model by introducing the additional constant term.

Examples. Let us look into two special cases of Gompertz-Mekeham law

(i) B = 0 (constant force of mortality, or exponential distribution)

In this case, $\mu_x = A$ hence, $\ell_x = \ell_0 e^{-Ax}$ and $e_x^\circ = \frac{1}{A}$

(ii)
$$A = 0$$
 (Gompertz)

In this case, $\ell_x = \ell_0 e^{-\frac{B}{\gamma}(e^{\gamma x}-1)}$ and $e_x^\circ = -\frac{1}{\gamma}e^P \operatorname{Ei}[-P]$ where $\operatorname{Ei}[z] = \int_{-\infty}^z \frac{e^x}{t} dt$, $P = \frac{Be^{\gamma x}}{\gamma}$.

Remark 1. More generally, we can consider: $\mu_x = \sum_{i=0}^m A_i x^i + \sum_{j=1}^n B_j e^{\gamma_j x}$, $(A_i, B_j, \gamma_j > 0)$

Exc. 2.4.2. In the above generalized model, derive ℓ_x for the following special cases.

(a) m = 1, n = 1 (Mekeham's second law).

(b) m = 0, n = 2 (Lazarus, Hardy)

Remark 2. As a mortality law applicable to whole range of age, Thiele (1871) proposed the following model.

$$\mu_x = a_1 e^{-b_1 x} + a_2 e^{-\frac{b_2}{2}(x-c)^2} + a_3 e^{b_3 x}, \quad (a_i, b_i, c > 0).$$

2.5 Random variable approach (*)

We have derived various probabilistic interpretations from the survival function. Taking the random variable as the starting point of the theory, we can have an equivalent theory, which we shall sketch in this section. This approach has an advantage of defining various concepts systematically.

The starting point of this theory is the assumption that the future lifetime of a newborn is a (continuous) random variable T distributed on $[0, \omega]$ with the cummulative distribution function (c.d.f.) $\Phi(t) = \Pr(T < t)$. In survival analysis it is convenient to work with the survival function (i.e., a complement of c.d.f. with respect to 1): $\tilde{\Phi}(t) := 1 - \Phi(t) = \Pr(T \ge t)$. The probability density function (p.d.f.) is defined by $\phi(t) := \Phi'(t)$. The meaning of p.d.f. is given by $\phi(t)dt = \Pr(t \le T < t + dt)$.

Force of mortality (alternatively called hazard ratio or inverse Mill's ratio) is defined by $\mu(t) := \frac{\phi(t)}{\tilde{\Phi}(t)} = -\frac{d}{dt} \log \tilde{\Phi}(t)$. The meaning of the force of mortality is given by $\mu(t)dt = \frac{\phi(t)dt}{\tilde{\Phi}(t)} = \frac{\Pr(t < T \leq t + dt)}{\Pr(T > t)} = \Pr(T < t + dt \mid T \geq t)$

To model the future lifetime of a life aged x > 0, we consider T - x given $T \ge x$, namely $T_x = (T - x) \mid (T \ge x)$. Put $T_0 = T$.

The basic quantities associated with T_x can be derived as follows.

$$\begin{split} \Phi(t \mid x) &= \Pr(T_x < t) = \Pr(T < x+t \mid T \geqq x) = \frac{\Pr(x \leqq T < x+t)}{\Pr(T \geqq x)} = 1 - \frac{\widetilde{\Phi}(x+t)}{\widetilde{\Phi}(x)}.\\ \widetilde{\Phi}(t \mid x) &= \Pr(T_x \geqq t) = \Pr(T \geqq x+t \mid T \geqq x) = \frac{\Pr(T \geqq x+t)}{\Pr(T \geqq x)} = \frac{\widetilde{\Phi}(x+t)}{\widetilde{\Phi}(x)}.\\ \phi(t \mid x) &= \frac{\partial}{\partial t} \Phi(t \mid x) = -\frac{\partial}{\partial t} \widetilde{\Phi}(t \mid x) = \frac{\phi(x+t)}{\widetilde{\Phi}(x)}.\\ \mu(t \mid x) &= \frac{\phi(t \mid x)}{\widetilde{\Phi}(t \mid x)} = -\frac{\partial}{\partial t} \log \widetilde{\Phi}(t \mid x). \end{split}$$

Equipped with these notions, we can now derive some important relations.

$$\widetilde{\Phi}(t+u \mid x) = \widetilde{\Phi}(u \mid x+t) \cdot \widetilde{\Phi}(t \mid x).$$
(1)

This equation implies

$$\Pr(T \ge x + t + u \mid T \ge x) = \Pr(T \ge x + t + u \mid T \ge x + t) \cdot \Pr(T \ge x + t \mid T \ge x).$$
(1)

$$\mu(t \mid x) = \mu(x+t) \qquad \left(check \qquad \frac{\phi(t \mid x)}{\widetilde{\Phi}(t \mid x)} = \frac{\phi(x+t)}{\widetilde{\Phi}(x+t)} \right). \tag{2}$$

From (2) and the definition of the force of mortality,

$$\frac{\partial}{\partial t}\widetilde{\Phi}(t \mid x) = -\mu(x+t) \cdot \widetilde{\Phi}(t \mid x)$$
or
$$\frac{\partial}{\partial t}\widetilde{\Phi}(t) = -\mu(t) \cdot \widetilde{\Phi}(t).$$
(3)

By integrating (3) with respect to t with the initial condition $\widetilde{\Phi}(t \mid x) \mid_{t=0} = 1$,

$$\widetilde{\Phi}(t \mid x) = \exp\left(-\int_0^t \mu(x+s)ds\right).$$
(4)

Putting t = 0 in (3) and using $\widetilde{\Phi}(t \mid x) \mid_{t=0} = 1$, we have

$$\mu(x) = -\frac{\partial}{\partial t} \widetilde{\Phi}(t \mid x) \mid_{t=0}$$
(5)

$$\mu(x)dt = -\widetilde{\Phi}(dt \mid x) = \Phi(dt \mid x) \ \left(= \Pr(T < x + dt \mid T \ge x)\right).$$
(5)

By operating $\frac{\partial}{\partial t} \cdot |_{t=0}$ on (1) and using (3) (or alternatively, by differentiating (4)),

$$\frac{\partial}{\partial x}\widetilde{\Phi}(t\mid x) = (\mu(x) - \mu(x+t)) \cdot \widetilde{\Phi}(t\mid x).$$
(6)

From the relation of c.d.f. and p.d.f., definition of mortality force, (2) and (5)

$$\widetilde{\Phi}(t \mid x) = \int_{t}^{\infty} \phi(s \mid x) ds = \int_{t}^{\infty} \mu(x+s) \cdot \widetilde{\Phi}(s \mid x) ds = \int_{t}^{\infty} \Phi(ds \mid x+s) \cdot \widetilde{\Phi}(s \mid x).$$
(7)

This relation implies

$$\Pr\left(T \ge x+t \mid T \ge x\right)$$

$$= \int_{t}^{\infty} \Pr(T < x+s+ds \mid T \ge x+s) \cdot \Pr(T \ge x+s \mid T \ge x)$$

$$= \int_{t}^{\infty} \Pr(x+s \le T < x+s+ds \mid T \ge x).$$
(7)

Exc. 2.5.1. Put $P(x \mid x + t) = \widetilde{\Phi}(t \mid x)$. Discuss the similarity between survival model and interest model by comparing $P(x \mid x + t)$ and $W(x \mid x + t)$; and, $\mu(x)$ and $\rho(t)$, respectively.

The life expectancy at age x > 0 is defined as the mean of T_x ,

$$e_x^\circ = E[T_x] = E[T - x \mid T \ge x] = \int_0^\infty t\phi(t \mid x) \, dt = \int_0^\infty \widetilde{\Phi}(t \mid x) \, dt.$$

The median future lifetime med(x) is defined as the age such that $\widetilde{\Phi}(med(x) \mid x) = \frac{1}{2}$.

Exc. 2.5.2. Write down the formula for $Var[T_x]$. (Just write down. You don't need to simplify your result.)

Exc. 2.5.3. For the random variable T, we define the limited expected value function E[T;x] by $E[T;x] = \int_0^x t\phi(t) dt + x \tilde{\Phi}(x)$. Namely, consider the following random variable which is a cut-off of T at $x: T \wedge x = min(T,x)$ then put $E[T;x] = E[T \wedge x]$. Show that E[T;x] and e_x° are related through $E[T] = E[T;x] + e_x^\circ \tilde{\Phi}(x)$.

2.6 Stochastic process approach (*)

One of the drawbacks of the random variable approach is its inherent static feature. There is yet another approach which explicitly traces the transition of survival status along time. Define

$$\Lambda^{(x)}(t) = H(T_x - t) = \{ \begin{array}{cc} 1 & (if & T_x \ge t) \\ 0 & (if & T_x < t) \end{array} \}$$

An intuitive meaning of this binary random variable is the indicator of survival; that is, it observes the survival status of a life for every moment and returns the value 1 if alive, and 0 if dead.

Suppose T_x takes a realised value $T_x = s$, if we see the time development of $\Lambda^{(x)}(t)$ for a fixed s, we get a sample life path

$$\lambda_s(t) = \Lambda^{(x)}(t) \mid_{T_x = s} = H(s - t).$$

The distribution of $\Lambda^{(x)}(t)$ is determined by the following relations

$$\Pr\left(\Lambda^{(x)}(t)=1\right) = \Pr(T_x \ge t) = \Pr\left(\Lambda(x+t)=1 \mid \Lambda(x)=1\right) = \Pr(T \ge x+t \mid T \ge x).$$

The above basic quantity that characterises the stochaistic process $\Lambda^{(x)}(t)$ is called the transition probability and denoted by

$$P(x \mid x+t) = \Pr\left(\Lambda(x+t) = 1 \mid \Lambda(x) = 1\right).$$

Transition intensity is defined by

$$\mu(x) = -\lim_{t \to 0} \frac{P(x \mid x+t) - P(x \mid x)}{t} = -\frac{\partial}{\partial t} P(x \mid x+t) \mid_{t=0} = -D_2 P(x \mid x) = -P_2(x \mid x).$$

Exc. 2.6.1. Show that the above defined $\mu(x)$ is identical to the force of mortality as defined earlier. In summary, we have, for a fixed time t,

$$E_{T_x}[\lambda_{T_x}(t)] = \int_0^\infty \lambda_s(t)\phi(s \mid x) \, ds = \int_t^\infty \phi(s \mid x) ds = \widetilde{\Phi}(t \mid x)$$
$$= \Pr(T_x \ge t) = \Pr\left(\Lambda^{(x)}(t) = 1\right) = \frac{\ell_{x+t}}{\ell_x}.$$

Exc. 2.6.2(*). Applying the Law of Large Numbers to the left-hand side of the above equation and discuss the implication to the relation between sample life path and the survival function.

Remark. Mathematically, $\{\Lambda^{(x)}(t)\}_{t\in[0,\infty)}$ is a 2-state Markov process with state 0 as the absorbing state. We can, if we wish, reconstruct an equivalent theory by using only the properties of this Markov process. For instance, T_x is constructed from this process by

$$T_x = \max\left\{t \mid \Lambda^{(x)}(t) = 1\right\}.$$

The relation

$$P(x \mid x + t + u) = P(x + t + u \mid x + t) \cdot P(x + t \mid x)$$

can be shown as follows:

$$\begin{split} &\Pr\left(\Lambda(x+t+u)=1\mid\Lambda(x)=1\right)\\ &= \begin{array}{l} \Pr(\Lambda(x+t+u)=1\cap\Lambda(x+t)=1\mid\Lambda(x)=1)\\ &+\Pr(\Lambda(x+t+u)=1\cap\Lambda(x+t)=0\mid\Lambda(x)=1)\\ &=\Pr(\Lambda(x+t+u)=1\mid\Lambda(x+t)=1\cap\Lambda(x)=1)\cdot\Pr(\Lambda(x+t)=1\mid\Lambda(x)=1)\\ &=\Pr(\Lambda(x+t+u)=1\mid\Lambda(x+t)=1)\cdot\Pr(\Lambda(x+t)=1\mid\Lambda(x)=1). \end{split}$$

A theoretical advantage of this approach is that it is easy to generalize to multi-state model with more than two states.

Exc. 2.6.3. From a point of view of social insurance, list up possible status of a worker and consider the transition between these states (draw the transition graph). What data are necessary to carry out calculation and how can one estimate them?

2.7 Life table functions

So far we have assumed the existance of smooth survival functions. In practice, however, one cannot expect to have such functions. A conventional and useful way is to tabulate these functions at integer values of x.

$$\ell_0 > \ell_1 > \ell_2 > \ell_3 > \dots > \ell_{\omega-1} > \ell_\omega = 0.$$

A life table presents these data and values of associated functions:

$$\begin{split} & d_x = \ell_x - \ell_{x+1}, \\ & q_x = \frac{d_x}{\ell_x} = 1 - \frac{\ell_{x+1}}{\ell_x}, \\ & p_x = 1 - q_x = \frac{\ell_{x+1}}{\ell_x}. \end{split}$$

Furthermore, we define

$$L_x = \int_x^{x+1} \ell_t dt \quad ; \quad T_x = \sum_{t=x}^{\infty} L_t = \int_x^{\infty} \ell_t dt \quad ; \quad m_x = \frac{d_x}{L_x} = \frac{\ell_x - \ell_{x+1}}{L_x}$$

Here m_x is called the central death rate (Note. The notation T_x was used earlier to denote a random variable. Do not confuse.)

Under the approximation

$$L_x = \int_x^{x+1} \ell_t dt \simeq \frac{\ell_x + \ell_{x+1}}{2},$$

we get

$$T_x = \sum_{t=0}^{\infty} L_{x+t} \simeq \frac{\ell_x}{2} + \sum_{t=1}^{\infty} \ell_{x+t} \quad ; \quad m_x = \frac{\ell_x - \ell_{x+1}}{L_x} \simeq 2\frac{\ell_x - \ell_{x+1}}{\ell_x + \ell_{x+1}} = 2\frac{1 - p_x}{1 + p_x} = \frac{q_x}{1 - \frac{1}{2}q_x}.$$

The curtate life expectancy is defined by

$$e_x = \frac{1}{\ell_x} \sum_{k=1}^{\infty} \ell_{x+k}.$$

This is related to the complete life expectancy by

$$e_x^\circ \simeq e_x + \frac{1}{2}.$$

Exc. 2.7.1. Verify this by using the Euler-Maclaurin summation formula.

In the random variable framework, consider the discretization of the continuous random variable T_x . Define

$$K_x = [T_x]$$

where the squre brackets denote the integer part, i.e. the nearest integer that does not exceed the value in the brackets. Then K_x is a discrete random variable, which takes values on $0, 1, 2, ..., [\omega - x]$.

From the life table, the probability distribution is derived by:

$$\Pr(K_x = k) = \Pr(k \le T_x < k+1) = \Pr\left(\Lambda^{(x)}(k+1) = 0 \mid \Lambda^{(x)}(k) = 1\right)$$

= $\tilde{\Phi}(k \mid x) - \tilde{\Phi}(k+1 \mid x) = \Phi(1 \mid x+k) \cdot \tilde{\Phi}(k \mid x)$
= $q_{x+k} \cdot {}_k p_x = \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x}$

The curtate life expectancy that we defined above can have the following expression.

$$e_x = E[K_x] = \sum_{k=0}^{\infty} k \cdot \Pr(K_x = k) = \sum_{k=0}^{\infty} k \cdot q_{x+k} \cdot kp_x = \sum_{k=1}^{\infty} kp_x.$$

Exc. 2.7.2. Verify this relation.

2.8 UN model life tables and selection

In the absence of reliable life tables, we resort to some model tables. UN has developed model life tables for 5 regional patterns for a wide range of mortality level. The UN model life table used 8 parameter formula

$$q_x = A^{(x+B)^C} + D \exp(-E(\log x - \log F)^2) + \frac{GH^x}{1 + GH^x}$$

Usually we select tables with a limited number of clues such as life expectancy at birth (also available from UN).

Exc. 2.8.1. Choose a life table. Calculate life table functions.

Now suppose we have base population $\tilde{\ell}(x)$ and sample data of deaths for a specified period and we have chosen a model life table with $\tilde{q}(x)$. We see a typical statistical test of the model, called chi-square test of goodness of fit.

Given the age groups $(x_{k-1}, x_k]$ with $x_0 < x_1 < x_2 < \cdots < x_r$. With a sample of size n, let d_k^{obs} be the number of observations of deaths in age class $(x_{k-1}, x_k]$ and let $\tilde{d}_k = \frac{n}{\sum_{x=0}^{\infty} \tilde{\ell}_x} \sum_{x=x_{k-1}}^{x_k} \tilde{\ell}(x) \tilde{q}(x_k)$. Construct the chi-square statistic

$$T = \sum_{k=1}^{r} \frac{(d_k^{obs} - \widetilde{d}_k)^2}{\widetilde{d}_k}.$$

Reject the hypothesis that the model fits the data (null hypothesis H_0) for a large number of T. More precisely, we have

$$T \xrightarrow{p} \chi^2(r-1) \quad (as \ n \to \infty).$$

For a given significance level α , the critical value of chi-square distribution C_{α} such that $\Pr(T > C_{\alpha}) = \alpha$ under H_0 have been tabulated in any standard books on statistics.

2.9 Interpolation of life table functions for fractional ages, approximation of force of mortality

Life table functions are usually tabulated at integer ages only but we sometimes need to calculate these values at non-integer age for non-integer duration. The following is a list of assumptions to get such results. Let x be an integer and 0 < t < 1.

(i) Uniform distribution of deaths: $\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$. Or equivalently put as $tq_x = t \cdot q_x$. Under this assumption

$$_{t}q_{x+s} = \frac{t \cdot q_{x}}{1 - s \cdot q_{x}} \quad ; \quad \mu_{x+t} = \frac{q_{x}}{1 - t \cdot q_{x}}.$$

(ii) Constant force of mortality: $\mu_{x+t} = \mu$. This is consistent with the exponential interpolation $\ln \ell_{x+t} = (1-t) \ln \ell_x + t \ln \ell_{x+1}$. Under this assumption

$$_{t}q_{x+s} = 1 - e^{-\mu t}$$
; $\mu_{x+t} = \mu$.

(iii) Balducci interpolation: $\frac{1}{\ell_{x+t}} = \frac{1-t}{\ell_x} + \frac{t}{\ell_{x+1}}$.

$$_{t}q_{x+s} = \frac{t \cdot q_{x}}{1 - (1 - t - s)q_{x}}$$
; $\mu_{x+t} = \frac{q_{x}}{1 - (1 - t)q_{x}}.$

(iv) Other interpolation methods

One can apply polynomial interpolation which smoothly (i.e. up to the first derivative) links the curve with the next age intervals on both sides. One of the simplest formula is King-Karup interpolation formula:

$$\ell_{x+t} = \ell_x + t\Delta\ell_{x-1} + \frac{1}{2}t(t+1)\Delta^2\ell_{x-1} + \frac{1}{2}t^2(t-1)\Delta^3\ell_{x-1}.$$

Where Δ is the forward difference operator $\Delta \ell_x = \ell_{x+1} - \ell_x$. Note that this assumption requires not only values ℓ_x, ℓ_{x+1} but also $\ell_{x-1}, \ell_x, \ell_{x+1}, \ell_{x+2}$. approximation. Since this is a polynomial of t, we can calculate μ_{x+t} with no difficulty.

Next, we consider the estimation of the force of mortality from life table functions. From Taylor series expansion,

$$\ell(x \pm h) = \ell(x) \pm \ell'(x)h + \frac{1}{2!}\ell''(x)h^2 \pm \frac{1}{3!}\ell'''(x)h^3 + \cdots,$$

we derive the following approximations (by truncation of second and third order respectively):

$$\mu_x = -\frac{1}{\ell_x} \frac{d\ell_x}{dx} \simeq \frac{\ell_{x-1} - \ell_{x+1}}{2\ell_x},$$
$$\mu_x = -\frac{1}{\ell_x} \frac{d\ell_x}{dx} \simeq \frac{8(\ell_{x-1} - \ell_{x+1}) - (\ell_{x-2} - \ell_{x+2})}{12\ell_x}.$$

Note. These results can be obtained by differentiating Stirling's interpolation formula (see Mathematical notes).

Alternatively,

$$_{t}p_{x} = e^{-\int_{x}^{x+t} \mu_{s} ds}, \quad hence$$

 $\mu_{x+\frac{1}{2}} \simeq \int_{x}^{x+1} \mu_{s} ds = -\log p_{x} = \log(\frac{1}{1-q_{x}}) \simeq \frac{q_{x}}{1-\frac{q_{x}}{2}} = m_{x}$

3. LIFE ANNUITIES

In this section, we combine two building blocks - models of interest and survival - in the earlier sections and develop a theory of life annuities.

3.1 Some simple life annuities

A whole life annuity provides an annual payment of 1 unit at the beginning of each year as long as the beneficiary (of initial age x) is alive. Thus its expected present value is

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x.$$

By using the commutation function $D_x = v^x \ell_x$, we can rewrite the formula as follows:

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \,_k p_x = \sum_{k=0}^{\infty} v^k \, \frac{\ell_{x+k}}{\ell_x} = \sum_{k=0}^{\infty} \, \frac{v^{x+k} \ell_{x+k}}{v^x \ell_x} = \frac{1}{D_x} \sum_{k=0}^{\infty} \, D_{x+k}.$$

We can consider a whole life annuity as a contingent payment whose probability is given by the survival rates.

$$\ddot{a}_x = EPV\left\{(1, k, \, _kp_x) \mid k = 0, 1, 2, \ldots\right\} = \sum_{k=0}^{\infty} v^k \, _k p_x.$$

We consider the following operations:

(i) *n*-year temporary life annuity

$$\ddot{a}_{x:\overline{n|}} = EPV\left\{(1,k,\,_k p_x) \mid 0 \le k \le n-1\right\} = \sum_{k=0}^{n-1} v^k\,_k p_x = \frac{1}{D_x} \sum_{k=0}^{n-1} D_{x+k}.$$

(ii) *n*-year deferred whole life annuity

$${}_{n|}\ddot{a}_{x} = EPV\left\{(1, k, {}_{k}p_{x}) \mid n \leq k < \infty\right\} = \sum_{k=n}^{\infty} v^{k} {}_{k}p_{x} = \frac{1}{D_{x}} \sum_{k=n}^{\infty} D_{x+k}.$$

(iii) Whole life annuity with *n*-year term-certain (guarantee period)

In this case, the contingent probability is given by $\pi_k = \{ \begin{array}{cc} 1 & (0 \leq k \leq n-1) \\ _k p_x & (k \geq n) \end{array}$

Thus

$$\begin{split} & EPV\left\{(1,k,\,\pi_k) \mid k=0,1,2,\ldots\right\} \\ & = EPV\left\{(1,k,1) \mid 0 \leq k \leq n-1\right\} + EPV\left\{(1,k,\,_kp_x) \mid n \leq k < \infty\right\} \\ & = \ddot{a}_{\overline{n|}} + {}_{n|} \ddot{a}_x \end{split}$$

Exc. 3.1.1. Discuss the sensitivity of \ddot{a}_x with respect to x and i. (qualitatively)

Exc. 3.1.2. Let r be the pension age. For x < r, the value $C_x^{UC} =_{r-x|} \ddot{a}_x$ correspond to what situation? Show that $C_x^{UC} < C_{x+1}^{UC}$ what is the intuition behind this fact? C_x^{UC} is called the unit cost contribution rate (for age x).

Exc. 3.1.3. Show that

$$\ddot{a}_x = \ddot{a}_{x:\overline{n|}} +_{n|} \ddot{a}_x.$$

$$\ddot{a}_x = 1 + a_x.$$

Exc. 3.1.4. Define the increasing annuity by $(Ia)_x = \sum_{k=1}^{\infty} k v^k {}_k p_x$. Show $(Ia)_x = \sum_{k=0}^{\infty} {}_{k|} a_x$. Derive the formula using D_x .

Exc. 3.1.5. Show that $\frac{d}{dv}\ddot{a}_x = \frac{1}{v}(Ia)_x$. Evaluate $\frac{d}{di}\ddot{a}_x$.

Exc. 3.1.6. Construct a spreadsheet model for calculating these types of life annuities.

3.2 Applications to pensions

(i) Annuitisation

Suppose a worker retired at age x convert the balance in his/her retirement savings (or in the provident fund) into a whole life annuity. Let us calculate the (non-indexed) annual pension she gets.

$$p \cdot \ddot{a}_x = Bal$$
 $\therefore p = \frac{Bal}{\ddot{a}_x}.$

Example. Fiji NPF. China Individual Account.

(ii) Contribution rate under the entry-age normal method (entry-age actuarial contribution)

Under this method, the projected benefit is funded by a level contribution from entry age (x) until retirement age (r). By equating the expected present values of contributions and benefits, we have

$$C^{EAN} \cdot \ddot{a}_{x:\overline{r-x|}} = \ _{r-x|} \ddot{a}_x \quad \therefore C^{EAN} = \frac{r-x|\ddot{a}_x}{\ddot{a}_{x:\overline{r-x|}}}.$$

Exc. 3.2.1. Discuss the sensitivity of C^{EAN} with respect to x and r.

(iii) Reduction rate for early retirement

Suppose the normal retirement age is r. If a worker opts for an early retirement pension at age r - n. To make no difference between early retirement and normal retirement, a reduction needs to be applied to the full pension in case of early retirement. Let ρ be the reduction rate per year of anticipation

$$(1-\rho)^n \cdot \ddot{a}_{r-n} = {}_n |\ddot{a}_{r-n} \qquad \therefore \rho = 1 - \sqrt[n]{\frac{n|\ddot{a}_{r-n}}{\ddot{a}_{r-n}}}$$

Exc. 3.2.2. Evaluate ρ . Example. The case of Germany, Japan.

(iv) Conversion of the balance in the savings account into credit

Suppose a provident fund is converted into defined-benefit social insurance pension with the following pension formula: Pension = (accrual rate) × (indexed career average salary) × (year of contribution) = aST. Given the accrued rate, how can a given balance be converted into contribution year? Assume that the indexed career average salary is estimated by the current salary and expected wage increase β . Then

$$aTS_x(1+\beta)^{r-n} \cdot_{r-x} \ddot{a}_x = Bal \qquad \therefore T = \frac{Bal}{aS_x(1+\beta)^{r-n} \cdot_{r-x} \ddot{a}_x}$$

Exc. 3.2.3. Discuss the advantages and disadvantages of this formula.

Exc. 3.2.4. The following formula is applied in some situation. Discuss the derivation of this formula and its financial implications. Compare with the above result.

$$T_1 = \frac{Bal}{(PV \ contribution \ rate) \times (Salary \ at \ conversion)}$$

3.3 Annuity payable more frequently than once a year

Suppose we divide one year equally into m intervals (each has a length of $\frac{1}{m}$) where m is an integer > 1. Consider a life annuity of 1 per annum, for a life aged x, with $\frac{1}{m}$ payable at the beginning of each m-thly period. The expected present value of this annuity, denoted $\ddot{a}_x^{(m)}$, is given by:

$$\ddot{a}_x^{(m)} = \sum_{k=0}^\infty \frac{1}{m} v^{\frac{k}{m}} \, _{\frac{k}{m}} p_x = \frac{1}{m\ell_x} \sum_{k=0}^\infty \ell_{x+\frac{k}{m}} v^{\frac{k}{m}} = \frac{1}{mD_x} \sum_{k=0}^\infty D_{x+\frac{k}{m}}.$$

Applying the Woolhouse summation formula, we have

$$\frac{1}{mD_x}\sum_{k=0}^{\infty} D_{x+\frac{k}{m}} = \frac{1}{D_x} \left[\sum_{k=0}^{\infty} D_{x+k} - \frac{m-1}{2m} \left(D_x + D_\infty \right) - \frac{1}{12} \frac{m^2 - 1}{m^2} \left(D'_\infty - D'_x \right) + \cdots \right].$$

Neglecting the terms of second or higher, and noting that $D'_x = -D_x (\mu_x + \rho)$, we have

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2 - 1}{12m^2}(\mu_x + \rho).$$

In practice, the following simple formula gives a good enough approximation:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$$

Exc. 3.3.0. Check that $D'_{x} = -D_{x} (\mu_{x} + \rho)$.

Exc. 3.3.1. Derive the above simple formula for $\ddot{a}_x^{(m)}$ by making the following linear approximation:

$$D_{x+t+\frac{k}{m}} = D_{x+t} + \frac{k}{m}(D_{x+t+1} - D_{x+t}).$$

Exc. 3.3.2. Show that

$$a_x^{(m)} = a_x + \frac{m-1}{2m}.$$

3.4 Continuous annuity

This is a continuous analogue of discrete life annuity. The formula is given by

$$\overline{a}_x = \int_0^\infty {}_t p_x v^t dt = \frac{1}{D_x} \int_0^\infty D_{x+t} dt.$$

Substituting

$$_{t}p_{x} = \exp\left(-\int_{0}^{t}\mu_{x+s}ds\right) \quad ; \quad v = e^{-\rho},$$

we have

$$\overline{a}_x = \int_0^\infty \exp\left(-\int_0^t (\rho + \mu_{x+s})ds\right) dt.$$

Alternatively, we can define

$$\overline{a}_x = \lim_{m \to \infty} \ddot{a}_x^{(m)} \simeq \ddot{a}_x - \frac{1}{2}$$
$$= \lim_{m \to \infty} a_x^{(m)} \simeq a_x + \frac{1}{2}.$$

Example. Under the Gompertz-Makeham law $\mu_x = A + Be^{\gamma x}$,

$$\overline{a}_x = \int_0^\infty \exp\left(-\int_0^t (\rho + A + Be^{\gamma(x+s)})ds\right) dt.$$

We see

$$\frac{\partial}{\partial A}\overline{a}_x = \frac{\partial}{\partial\rho}\overline{a}_x.$$

Exc. 3.4.1. Interpret the above relation. What about the sensitivity with respect to B?

Exc. 3.4.2. Calculate \bar{a}_x under the Gomperz-Makeham law $\mu_x = A + Be^{\gamma x}$. [Use the results in 2.4 (formula of e_x° under G-M) with $\mu_x = \rho + A + Be^{\gamma x}$]

3.5 Complete annuity and its refinement

(1) Complete annuity

In reality, there is some time difference between the last full payment of annuity and the time of death. We consider a type of life annuity, called the complete annuity, that also pays a final fractional payment that takes into account this time difference. The provision is as follows:

(i) If the beneficiary is alive at the end of each *m*-thly period, $\frac{1}{m}$ is payable.

(ii) At the death of the beneficiary, a final payment which is proportial to the time elapsed since the last full payment is paid. Note that we have already known the first payment (whole life annuity payable m times at the end of each period). The expected present value of the complete annuity, denoted a_x° , is written as:

$$a_x^{\circ} = a_x^{(m)} + r_x^{(m)}.$$

The second term is calculated by

$$r_x^{(m)} = \frac{1}{\ell_x} \sum_{t=0}^{\infty} \sum_{i=0}^{m-1} \int_0^{\frac{1}{m}} \ell(x+t+\frac{i}{m}+\theta)\mu(x+t+\frac{i}{m}+\theta)\theta v^{t+\frac{i}{m}+\theta} d\theta$$

Using the approximation

$$\begin{split} \ell(x+t+\frac{i}{m}+\theta)\mu(x+t+\frac{i}{m}+\theta)v^{t+\frac{i}{m}+\theta} \\ &\simeq \ell(x+t+\frac{i}{m}+\frac{1}{2m})\mu(x+t+\frac{i}{m}+\frac{1}{2m})v^{t+\frac{i}{m}+\frac{1}{2m}} = v^{-x}D(x+t+\frac{2i+1}{2m})\mu(x+t+\frac{2i+1}{2m}), \end{split}$$

we have,

$$r_x^{(m)} = \frac{1}{D_x} \sum_{t=0}^{\infty} \sum_{i=0}^{m-1} D(x+t+\frac{2i+1}{2m}) \mu(x+t+\frac{2i+1}{2m}) \int_0^{\frac{1}{m}} \theta d\theta.$$

In the above,

$$\sum_{t=0}^{\infty} \sum_{i=0}^{m-1} D(x+t+\frac{2i+1}{2m})\mu(x+t+\frac{2i+1}{2m})$$

= $\sum_{i=0}^{\infty} D(x+\frac{2i+1}{2m})\mu(x+\frac{2i+1}{2m}) \simeq m \int_{0}^{\infty} D_{x+t}\mu_{x+t}dt = m\overline{M}_{x},$
$$\int_{0}^{\frac{1}{m}} \theta d\theta = \frac{1}{2m^{2}}.$$

Hence,

$$r_x^{(m)} \simeq \frac{1}{D_x} \cdot m \overline{M}_x \cdot \frac{1}{2m^2} = \frac{1}{2m} \frac{\overline{M}_x}{D_x}.$$

Therefore

$$a_x^{\circ} = a_x^{(m)} + r_x^{(m)} \simeq a_x + \frac{m-1}{2m} + \frac{1}{2m} \frac{M_x}{D_x}.$$

We use the following approximation

$$\overline{M}_x = \int_0^\infty D_{x+t} \mu_{x+t} dt \simeq \sum_{t=0}^\infty (\ell_{x+t} - \ell_{x+t+1}) v^{x+t+\frac{1}{2}} = \sum_{t=0}^\infty d_{x+t} v^{x+t+\frac{1}{2}}$$

(2) Complete annuity - a more general case

Let us further consider actual administrative constraint.

(i') We have assumed that payments are made according to exact age of beneficiaries, but the dates of payment are usually fixed in the calender (e.g. 1st of March, June, September, December). Thus, we assume that on average the birthday of beneficiary occurs at the middle of the payment period.

(ii') In case of the death of the beneficiary, a final payment which is proportial to the time elapsed since the last full payment is paid at the next payment period.

Let us calculate the present value of this life annuity, denoted by $\tilde{a}_x^{(m)}$. We assume that $1 \leq m \leq 12$ and *m* divides 12.

First, the annuity payment for living pensioners needs to be modified from $a_x^{(m)}$ to $\frac{1}{2m}|\ddot{a}_x^{(m)} \simeq \ddot{a}_x^{(m)} - \frac{1}{2m} \simeq \ddot{a}_x - \frac{1}{2}$. (Or from $a_x^{(m)}$ to \bar{a}_x)

Second, the expected present value of the final benefit for deceased pensioners is evaluated as follows:

$$\widetilde{r}_x^{(m)} = \frac{1}{\ell_x} \sum_{t=0}^{\infty} \sum_{i=0}^{m-1} \int_0^{\frac{1}{m}} \ell(x+t+\frac{i}{m}+\theta) \mu(x+t+\frac{i}{m}+\theta) \frac{[12\theta]_+}{12} v^{t+\frac{i+1}{m}-\frac{1}{2m}} d\theta.$$

where $[n]_+$ denotes the smallest integer that is greater or equal to n. In the above we use the approximation:

$$\begin{split} \ell(x+t+\frac{i}{m}+\theta)\mu(x+t+\frac{i}{m}+\theta)v^{t+\frac{i+1}{m}-\frac{1}{2m}} \\ \simeq \ell(x+t+\frac{i}{m}+\frac{1}{2m})\mu(x+t+\frac{i}{m}+\frac{1}{2m})v^{t+\frac{i}{m}-\frac{1}{2m}} = v^{-x}D(x+t+\frac{2i+1}{2m})\mu(x+t+\frac{2i+1}{2m}). \end{split}$$

Hence

$$\widetilde{r}_x^{(m)} = \frac{1}{D_x} \sum_{t=0}^{\infty} \sum_{i=0}^{m-1} D(x+t+\frac{2i+1}{2m}) \mu(x+t+\frac{2i+1}{2m}) \int_0^{\frac{1}{m}} \frac{[12\theta]_+}{12} d\theta$$

Since

$$\sum_{t=0}^{\infty} \sum_{i=0}^{m-1} D(x+t+\frac{2i+1}{2m})\mu(x+t+\frac{2i+1}{2m})$$
$$= \sum_{i=0}^{\infty} D(x+\frac{2i+1}{2m})\mu(x+\frac{2i+1}{2m}) \simeq m \int_{0}^{\infty} D_{x+t}\mu_{x+t}dt = m\overline{M}_{x},$$
$$\int_{0}^{\frac{1}{m}} \frac{[12\theta]_{+}}{12}d\theta = \frac{1}{(12)^{2}} \int_{0}^{k} [n]_{+}dn = \frac{1}{(12)^{2}} \sum_{n=1}^{k} n = \frac{1}{(12)^{2}} \cdot \frac{1}{2}k(k+1) \qquad (mk=12).$$

Hence,

$$\widetilde{r}_x^{(m)} = \frac{1}{D_x} \cdot m \overline{M}_x \cdot \frac{1}{(12)^2} \cdot \frac{1}{2} k(k+1) = \frac{k+1}{24} \cdot \frac{\overline{M}_x}{D_x} = \frac{m+12}{24m} \cdot \frac{\overline{M}_x}{D_x}.$$

Therefore,

$$\widetilde{a}_x^{(m)} =_{\frac{1}{2m}|} \ddot{a}_x^{(m)} + \widetilde{r}_x^{(m)} \simeq \ddot{a}_x - \frac{1}{2} + \frac{m+12}{24m} \cdot \frac{\overline{M}_x}{D_x}$$

Note. Another derivation of $\tilde{r}_x^{(m)}$. Let mk = 12.

$$\tilde{r}_x^{(m)} = \frac{1}{\ell_x} \sum_{t=0}^{\infty} \left[\sum_{\mu=0}^{m-1} \sum_{\lambda=1}^k \left(\ell(x+t+\frac{k\mu+\lambda}{12}) - \ell(x+t+\frac{k\mu+\lambda+1}{12}) \right) \frac{\lambda}{12} v^{t+\frac{k(\mu+1)}{12}} \right].$$

In the above,

$$\ell(x+t+\frac{k\mu+\lambda}{12}) - \ell(x+t+\frac{k\mu+\lambda+1}{12}) \simeq \frac{1}{12} \left(\ell_{x+t} - \ell_{x+t+1}\right) = \frac{d_{x+t}}{12}.$$

Hence,

$$\begin{split} \widetilde{r}_x^{(m)} &\simeq \frac{1}{\ell_x} \sum_{t=0}^{\infty} \sum_{\mu=0}^{m-1} \sum_{\lambda=1}^k \frac{d_{x+t}}{12} \frac{\lambda}{12} v^{t+\frac{k(\mu+1)}{12}} \\ &= \frac{1}{(12)^2} \left[\sum_{t=0}^{\infty} \frac{d_{x+t}}{\ell_x} \left(\sum_{\mu=0}^{m-1} v^{t+\frac{k(\mu+1)}{12}} \right) \right] \left(\sum_{\lambda=1}^k \lambda \right) \\ &\simeq \frac{1}{(12)^2} \left[\sum_{t=0}^{\infty} \frac{d_{x+t}}{\ell_x} \left(m v^{t+\frac{1}{2}} \right) \right] \frac{k(k+1)}{2} = \frac{k+1}{24} \frac{1}{D_x} \sum_{t=0}^{\infty} d_{x+t} v^{x+t+\frac{1}{2}}. \end{split}$$

3.6 Recursion formula, differential equation

In the discrete case, we start from

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \,_k p_x$$

By replacing $_{k}p_{x}$ by $_{k-1}p_{x+1} \cdot _{1}p_{x}$ we find

$$\ddot{a}_x = 1 + v\ddot{a}_{x+1}\,p_x.$$

The value of \ddot{a}_x can be calculated backward, starting with the condition at the limiting age $\ddot{a}_{\omega} = 0$. Exc. 3.6.1. Construct a spreadsheet model for this algorithm. Compare your results with those you did in 3.1.

In the continuous case, by differentiating

$$\overline{a}_x = \int_0^\infty \exp\left(-\int_0^t (\rho + \mu_{x+s}) ds\right) dt,$$

we find

$$\frac{d\overline{a}_x}{dx} = (\rho + \mu_x)\overline{a}_x - 1.$$

Conversely, by integrating the above differential equation with the boundary condition $\overline{a}_x = 0$ ($x \ge \omega$), we retrieve the above expression of \overline{a}_x .

Exc. 3.6.2. Verify this statement.

We can derive the above recursion formula and differential equation by considering the transition diagram (see figure).

3.7 Probabilistic approach (*)

For simplicity, we hereafter deal with the continuous annuity. However, most statements can be rephased for the discrete annuity. We start from

$$\overline{a}_x = \int_0^\infty e^{-\rho t} {}_t p_x \, dt = \int_0^\infty \overline{a}_{\overline{t|}} {}_t p_x \, \mu_{x+t} \, dt.$$

Exc. 3.7.1. Show this relation [hint: integration by parts]

First, note that

$$\overline{a}_x = \int_0^\infty \overline{a}_{\overline{t}|\ t} p_x \,\mu_{x+t} \, dt = \int_0^\infty \overline{a}_{\overline{t}|} \,\phi(t \mid x) dt = \int_0^\infty \overline{a}_{\overline{t}|} \Pr(t \leq T_x < t + dt).$$

This relation implies that if we define the random variable by $Y = \overline{a}_{\overline{T_x|}}$, then

$$\overline{a}_x = E[Y].$$

We can also derive further probabilistic characteristics of Y.

The cumulative distribution function (c.d.f.) of Y. By the transform of random variable (see Mathematical notes) , we get

$$F(y) = \Phi\left(-\frac{1}{\rho}\log(1-\rho y) \mid x\right)$$
$$= \frac{\ell(x-\frac{1}{\rho}\log(1-\rho y))}{\ell(x)}.$$

The probability density function (p.d.f.) of Y.

$$f(y) = F'(y) = \phi(-\frac{1}{\rho}\log(1-\rho y) \mid x)\frac{1}{1-\rho y} \\ = \frac{\ell(x-\frac{1}{\rho}\log(1-\rho y)) \ \mu(x-\frac{1}{\rho}\log(1-\rho y))}{\ell(x)}\frac{1}{1-\rho y}$$

The variance of Y. Recall

$$Var[Y] = E[Y^2] - (E[Y])^2 = E[Y^2] - (\overline{a}_x)^2.$$

Let

$$Y(t,\rho) = \overline{a}_{t|}(\rho) = \int_0^t e^{-\rho s} ds = \frac{1}{\rho} (1 - e^{-\rho t}).$$

Then

$$(Y(t,\rho))^{2} = \frac{1}{\rho^{2}} (1 - 2e^{-\rho t} + e^{-2\rho t}) = \frac{2}{\rho} (\frac{1 - e^{-\rho t}}{\rho} - \frac{1 - e^{-2\rho t}}{2\rho}) = \frac{2}{\rho} (Y(t,\rho) - Y(t,2\rho)).$$

Hence,

$$E[Y^2] = \frac{2}{\rho} (\overline{a}_x - \overline{a}_x).$$

Where $\overline{{}^2a_x}$ is the life annuity with double force of interest. Therefore,

$$Var[Y] = \frac{2}{\rho}(\overline{a}_x - \overline{a}_x) - (\overline{a}_x)^2.$$

Exc. 3.7.2. Under the assumption of a constant force of mortality, μ , and of a constant force of interest, ρ , evaluate

$$\overline{a}_x = E[Y]$$
; $Var[Y]$; $Pr(Y \ge \overline{a}_x)$.

Further derive these quantities under the Gompertz-Mekeham law.

Exc. 3.7.3. Evaluate Var[Y] numerically.

Alternatively, note that

$$\overline{a}_x = \int_0^\infty e^{-\rho t} {}_t p_x \ dt = \int_0^\infty e^{-\rho t} \widetilde{\Phi}(t \mid x) dt = \int_0^\infty e^{-\rho t} \Pr(T_x \ge t) dt = \int_0^\infty e^{-\rho t} \Pr(\Lambda_t^{(x)} = 1) dt.$$

This relation implies that if we define the stochastic process $\left\{\Lambda_t^{(x)}\right\}_{t\in[0,\infty)}$ by $\Lambda^{(x)}(t) = H(T_x - t)$, and put the function $Z(t) = E[\Lambda_t^{(x)}] = \Pr(\Lambda_t^{(x)} = 1)$, then

$$\overline{a}_x = PV[Z(t)].$$

The above two interpretations can be summarised by

$$\overline{a}_x = \int_0^\infty e^{-\rho t} \Pr(T_x \ge t) dt = \int_0^\infty e^{-\rho t} \Pr(\Lambda_t^{(x)} = 1) dt = \int_0^\infty \overline{a}_{\overline{t|}} \Pr(t \le T_x < t + dt),$$

or, put simply as

$$PV(\Pr(T_x \ge t)) = E[PV(\overline{a}_{\overline{T_x}|})]$$

Note that the same arguments run for the discrete case, we simply note the following:

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k|}} \cdot q_{x+k} \cdot {}_k p_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k|}} \cdot \Pr\left(K_x = k\right).$$

Exc 3.7.4(***). Suppose a contingent cash-flow is given by stochastic process $\{X_t\}_{t\in[0,\infty)}$. Consider the interest is also stochastic $\{\rho_t\}_{t\in[0,\infty)}$. Assume that X_t and ρ_s are stochastically independent for any t and s. Define a ramdom variable by $PV = \int_0^\infty X_t \exp(-\int_0^t \rho_s ds) dt$. Discuss the probabilistic properties of PV, e.g. E[PV], p.d.f. etc. [You need to consider the joint distributions of $\{X_t\}_{t\in[0,\infty)}$ and $\{\rho_t\}_{t\in[0,\infty)}$, which are functionals F(X) and $\Theta(\rho)$.] Further, what happens if X_t and ρ_s are dependent one another.

3.8 An inequality between life annuity and annuity certain

It is sometimes misunderstood that the life annuity \overline{a}_x is equal to the annuity certain $\overline{a}_{\overline{n|}}$ with $n = e_x^{\circ}$. These two values are different. In fact we have

$$\overline{a}_x < \overline{a}_{\overline{e_x^\circ}}$$

Mathematically, this is a direct consequence of Jensen's inequality, which states that for a random variable X and a concave function f > 0,

$$E[f(X)] \leq f(E[X]).$$

Exc. 3.8.1. Complete the proof of the above inequality. [Apply Jensen's inequality for $X = T_x$ and $f(t) = \overline{a_{t\bar{t}}}$. Show that f'' < 0]

Alternatively, we can show this result graphically (see figure).

3.9 Life insurance

A whole life insurance provides the 1 unit of payment at the time the insured (of initial age x) dies. We may regard this as the death benefit. In the continuous case, define

$$\overline{A}_{x} = \int_{0}^{\infty} v^{t} {}_{t} p_{x} \; \mu_{x+t} dt = \frac{1}{D_{x}} \int_{0}^{\infty} D_{x+t} \; \mu_{x+t} dt = \int_{0}^{\infty} \left[\mu_{x+t} \exp\left(-\int_{0}^{t} (\rho + \mu_{x+s}) ds\right) \right] dt$$

Note that

$$\overline{A}_x = \int_0^\infty e^{-\rho t} {}_t p_x \, \mu_{x+t} \, dt = \int_0^\infty e^{-\rho t} \, \phi(t \mid x) dt = \int_0^\infty e^{-\rho t} \, \Pr(t \le T_x < t + dt).$$

This relation implies that if we define the random variable by $Z = e^{-\rho T_x}$, then

$$\overline{A}_x = E[Z].$$

Exc. 3.9.1. Derive the probability density distribution of Z.

Exc. 3.9.2. Show that $a_x^{\circ} = a_x^{(m)} + \frac{1}{2m}\overline{A}_x$. Interpret.

Exc. 3.9.3. Show that \overline{A}_x satisfies the following differential equation. Interpret.

$$\frac{dA_x}{dx} = (\rho + \mu_x)\overline{A}_x - \mu_x.$$

From the definitions of \overline{a}_x and \overline{A}_x , we have the following relation:

$$\overline{A}_x = 1 - \rho \overline{a}_x$$

Exc. 3.9.4. Verify the above relation by integration by parts. Show that

$$\frac{d\overline{a}_x}{dx} = \mu_x \overline{a}_x - \overline{A}_x$$

Do these relations hold if interest is time dependent?

In the discrete time case, define

$$A_x = \sum_{k=0}^{\infty} \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x} v^{k+1} = \frac{1}{\ell_x} \sum_{k=0}^{\infty} d_{x+k} v^{k+1} = \frac{1}{D_x} \sum_{k=0}^{\infty} C_{x+k},$$

where $C_x = d_x v^{x+1}$.

Exc. 3.9.5. Show that

$$A_x = 1 - (1 - v)\ddot{a}_x = v\ddot{a}_x - a_x.$$

4. FINANCIAL PERFORMANCE INDICATORS AND THEIR IMPLICATIONS

One of the objectives of the actuarial valuation of social security pension schemes is to set out the future contribution rates which ensure the long-term solvency of the schemes. The requirements for the long-term solvency are, in many cases, formulated in terms of financial indicators. We examine the implications of three financial indicators and develop premium formulae which satisfy the required financial conditions.

4.1 Financial performance indicators

Let us consider a defined-benefit pay-as-you-go pension scheme whose resources come from contributions from workers' payroll, return on investment of the reserves, and (possibly) state subsidy. The financial operations of the fund are described by the following accounting identities:

$$R(t) = C(t) + I(t), (4.1.)$$

$$C(t) = p(t) \cdot S(t), \tag{4.2.}$$

$$\Delta F(t) = F(t) - F(t-1) = R(t) - B(t), \qquad (4.3.)$$

where,

- R(t) : the total revenue to the fund in year t,
- C(t): the total contributions collected during the year t,
- p(t) : the contribution rate in year t,
- S(t) : the aggregate contributory salaries in year t,
- I(t) : the investment income in year t,
- B(t) : the total expenditure of the scheme in year t,
- F(t) : the reserve of the fund at the end of year t.

Standardized financial performance indicators are useful for analysing the long-term transition of the financial status and comparing financial positions of different schemes, rather than the expenditure and total contributory earnings expressed in nominal terms. We define three financial performance indicators of a pension scheme, called the pay-as-you-go cost rate, the reserve ratio and the balance ratio.

Firstly, the PAYG cost rate in year t is defined as

$$C^{PAYG}(t) = \frac{B(t)}{S(t)}.$$
(4.4.)

This is conceived as the contribution rate needed for payment of expenditure in the current year if costs were to be financed solely from the current contributory earnings. The PAYG cost rate is further expressed as a product of two factors:

$$C^{PAYG}(t) = d(t) \cdot \theta(t). \tag{4.5.}$$

Here, d(t) is the demographic dependency ratio, defined by the number of pensioners as a percentage of the covered population; and, $\theta(t)$ is the average replacement ratio, defined by the average pension as a percentage of the average contributory earnings. This relation implies that if the relative number of active contributors to pensioners is lower or the relative pension level as compared with the average contributory earnings is higher, then the PAYG cost rate is higher. If the replacement ratio does not change significantly over time, the demographic ratio is considered to be the determinant of the long-term development of the PAYG cost rate. A rapid population ageing may cause a rapid increase in the PAYG cost rate.

Secondly, the funding ratio (alternatively, reserve ratio) in year t is defined as

$$a(t) = \frac{F(t-1)}{B(t)},$$
(4.6.)

which represents the relative level of the reserve as a multiple of annual expenditure. It represents how many years the fund (held at the beginning of a year) would be able to pay the current level of expenditure by liquidating the reserve (assuming that the reserve were fully liquidable). The funding ratio may be considered as an indicator of the short-term solvency of the scheme. For example, if pensions are paid monthly, the scheme should retain a reserve at least equal to one month's benefit at the beginning of each month, which is approximately 9 % of the annual expenditure. It is worth remembering that the minimum level of the funding ratio as a working capital is 0.09.

Thirdly, the balance ratio in year t is defined as

$$b(t) = \frac{B(t) - C(t)}{I(t)}.$$
(4.7.)

This indicator represents the amount of expenditure in excess of the income from contributions and state subsidy expressed as a percentage of the expected interest income. To clarify the meaning of this indicator, we first note the liquidity of major income available to the fund. While contributions (and the state subsidy) are equivalent to cash, the liquidity of the investment income generally depends on the types of assets in investment. Further, the reserve can be used to cover the expenditure. Usually, special arrangements are necessary to liquidate the assets in investment.

The balance ratio characterizes the status of the fund's balance in the following manner. Suppose a newly implemented pension scheme whose expenditure is initially small but gradually increases at a rate faster than the contributions and eventually exceeds them. In Stage I, where there is sufficient cash inflow to cover the expenditure (i.e. the contributions exceed the expenditure), the balance ratio is less than 0. In Stage II, where cash income from contributions is no longer sufficient to cover the expenditure but the balance is positive if the investment income is taken into account, the balance ratio takes values between 0 and 1. In this case the value of balance ratio represents the percentage of investment income that needs to be liquidated in order to finance the amount of expenditure in excess of the total cash income. Finally, in Stage III, the total revenue is less than the expenditure and the deficit in the balance should be met by the liquidation of the reserve. In this case, the balance ratio is greater than 1. The following table summarises the relation between the cash flow and reserve in terms of the balance ratio.

Balance ratio	Balance	Cash-flow	Reserve	Financial status
b(t) < 0	B(t) < C(t)	positive	increase	contributions exceed expenditure;
0(0) < 0	D(0) < C(0)	positive	ther case	no need to liquidate interest income.
b(t) = 0	B(t) - C(t)	~0r0	increase	contributions just equal expenditure;
b(t) = 0	D(t) = C(t)	2010	increase	the fund is in pay-as-you-go state.
				contributions are less than expenditure
0 < b(t) < 1	C(t) < B(t) < R(t)	negative	increase	however, with interest income,
				the balance is positive (in surplus).
1(1) = 1	$\mathbf{D}(\mathbf{i}) = \mathbf{D}(\mathbf{i})$, .		contributions and interest income
b(t) = 1	B(t) = R(t)	negative	stationary	equal expenditure; the balance is zero.
	$\mathbf{D}(t) = \mathbf{D}(t)$			total income is less than expenditure;
b(t) > 1	B(t) > R(t) ($\Delta F(t) < 0$)	negative	decrease	the balance is negative (in deficit)
				and reserve is declining.

Note that a negative balance ratio means a positive cash-flow (i.e. contributions less expenditure) to the scheme, and that a positive balance ratio means a negative cash-flow to the scheme.

Alternatively, the relation between the cash flow and reserve of a pension fund can be described by using the phase diagram . If we denote by i the annual average rate of return on the investment of the reserve (for simplicity, i is assumed to be constant over time), then the investment income is written as

$$I(t) = i \cdot F(t) + h \cdot D(t), \qquad (4.8.)$$

where D(t) = C(t) - B(t) is the cash inflow to the fund and $h = \sqrt{1+i} - 1 \simeq \frac{i}{2}$.

Substituting (4) into (3), we obtain the following difference equation describing the evolution of the reserve:

$$\Delta F(t) = \sqrt{1+i}D(t) + i \cdot F(t). \tag{4.9.}$$

Then, each point in the upper half domain in the D-F plane represents a financial position of the fund . The curve drawn in Figure represents the development path of the pension scheme in the above example. As illustrated in the Figure, the whole upper plane can be divided into three regions according to the value of the balance ratio. In the above terminology, Stage I corresponds to the north-east quadrant in which both F and D are positive. Stage II corresponds to the area between the vertical axis and the downward sloping line

(L):
$$F = -\frac{\sqrt{1+i}}{i} \cdot D.$$
 (4.10.)

Here, (L) represents a critical line on which the marginal increase in the reserve is zero. Stage III corresponds to the region left to the line (L).

Exc. Study the financial implication of this indicator.

4.2 Examples of the use of financial performance indicators in selected countries

In this section, we illustrate how financial indicators are actually used in the financing of existing social security pension schemes. For this purpose, four OECD countries have selected, namely the USA, Germany, Japan and Canada.

Firstly, under the Old-Age, Survivors and Disability Insurance (OASDI) of the United States, benefits are adjusted according to changes in the consumer price index (CPI). However, there is a provision which states that if the reserve ratio, named the "OASDI trust fund ratio", is less than 20.0 % at the beginning of the year, then the cost-of-living adjustment in benefits in that year will be limited to the CPI increase or the wage increase, whichever is the lower. In addition, the Trustee Board of the OASDI Trust Fund has adopted a short-range test of financial adequacy which requires the "OASDI trust fund ratio" to be at least equal to 1.0 throughout the next ten years.

Secondly, under the Employee's Mandatory Pension Insurance of Germany, the contribution rate is determined every year so that the liquid reserve at the end of the year should be at least equal to one month of estimated annual expenditure excluding the Federal subsidy. This condition can be expressed such that the reserve ratio should be more than one-twelfth, i.e. $a(t) \geq 0.09$.

Thirdly, according to the 1999 actuarial valuation of the Employee's Pension Insurance of Japan, the future contribution rate is determined by the following conditions:

1° The contribution rate is raised every five years at a constant rate until a certain target year.

 2° From the target year, an actuarial level premium, called the ultimate contribution rate, is applied. The target year is chosen so that the ultimate rate is contained within 20 percent of gross earnings.

 3° The scheme secures liquid income sufficient to meet the benefit payment in each year.

 4° A certain amount of reserve is set up for unanticipated shocks.

Conditions 1° and 2° lead to the following formula for the future contribution rate:

$$p(t) = p_0 + \Delta p([\frac{t - t_0}{5}] + 1) \qquad (for \quad t_0 \le t \le T - 1)$$

= p_{\max} $(for \quad t \ge T),$

where

 t_0 : Base year of valuation.

 p_0 : Contribution rate in the base year.

 p_{\max} : Ultimate contribution rate.

 Δp : Step of increase in the contribution rate for every 5 years.

T: Target year of the ultimate contribution rate (assumed to be 2025).

[x] : The integer part of x.

Conditions 3° and 4° can be interpreted as $b(t) \leq b_0$, and $a(t) \geq a_0$, where b_0 and a_0 are the required level of balance ration and (liquid) reserve ratio, respectively.

Fourthly, in the 15th actuarial valuation of the Canada Pension Plan, the so-called "15-year formula" is applied to determine the contribution rates for the period after 2016. This formula is described as follows:

1° The contribution rate is raised every year at a constant rate.

2° The annual rate of increase of the contribution rate is revised every five years.

 3° The rate of increase is determined as the lowest rate of increase such that, if it were applied for the next 15 years, the expected reserves at the end of this period would be at least equal to twice the expenditure in the subsequent year.

From condition 1°, the formula for the contribution rates after 2016 is

$$p(t) = p_i + \Delta p_i \cdot (t - t_i)$$
 (for $t = t_i + 1, ..., t_i + 5$; $t_i = t_0 + 5i$, $i = 0, 1, 2...$)

where,

 t_i : Year of i-th contribution review after 2016 (recurring every five years).

 p_i : Contribution rate in year t_i .

 Δp_i :Annual rate of increase in the contribution rate for $t_i + 1 \le t \le t_i + 5$.

For each year of contribution review at t_i , the rate of increase applied for the next 5 years, Δp_i , is determined in accordance with the condition 3°, i.e. $a(t_i + 15) \ge 2$. To estimate $a(t_i + 15)$, it has been assumed that the contribution rate is raised continuously at the constant annual rate Δp_i until $t = t_i + 15$.

The above-explained provision concerning the schedule of future contribution rates has been changed in the subsequent amendments of the Act. According to the reports of 16th and 17th actuarial valuations, the following methods have been applied to set the long-term contribution rates.

 1° The schedule of increase in contribution rates until 2002 is prescribed and no subsequent increase is scheduled.

2° For the period in and after 2003, a level contribution rate, called the "steady-state contribution rate" is applied.

3° The "steady-state contribution rate" is determined as the lowest constant rate which will enable for the reserve ratio to remain generally constant.

In condition 3° , the interpretation of the requirement of "generally constant" reserve ratio is left to the judgement of actuaries and may be redetermined for each valuation. In fact, the comparison of the reserve ratios in 2030 and 2100 was adopted (i.e. a(2030) = a(2100)) in the 16th valuation. In the 17th valuation, however, the reserve ratios in 2010 and 2060 were chosen for this purpose (i.e. a(2010) = a(2060). The projection results show that a(2010) = 4.12 and a(2060) = 5.38 under the standard set of assumptions.).

5. METHODS OF EVALUATING THE FINANCIAL SUSTAINABILITY OF PEN-SION SCHEMES

5.1. Long-term solvency condition and liquidity requirement

To examine the sustainability of a defined-benefit pension scheme from the point of view of the financial manager, two conditions should be taken into account. One is the long-term solvency on a discounted present value basis. The other is the liquidity requirement for the benefit payment.

The condition of the long-term solvency of a pension fund is stated by the following equation.

$$PV[C(t)] + F_0 = PV[B(t)].$$
(5.1.)

Here C(t) is the stream of projected contribution income, F_0 is the reserve held on the valuation date, and E(t) is the stream of projected expenditure; $PV[\cdot]$ is the operation taking the present value. Note that from this condition it follows that

$$\lim_{t \to \infty} PV[F(t)] = 0. \tag{5.2.}$$

(Exc. Show the above equation.)

The difference between the left-hand side and right-hand side in the solvency condition (5.1.) is called "the actuarial balance". If a negative actuarial balance is detected it is called "the unfunded pension liability" or "the implicit pension debt". Namely,

$$UPL (= IPD) = PV[B(t)] - PV[C(t)] - F_0.$$
(5.3.)

To test the long-term solvency, a suitable methodological tool is the actuarial balance sheet, which looks as follows:

Assets		Liabilities	
- Reserves (a.1.)	F_0	- Benefits (b)	$\sum_{t=1}^{\infty} \frac{E(t)}{(1+i)^t}$
- Contributions (a.2.)	$\sum_{t=1}^{\infty} \frac{C(t)}{(1+i)^t}$		
	. ,	- Actuarial balance	(a.1.)+(a.2.)-(b)

The actuarial balance sheet compares the sum of the reserves at hands F_0 (item a.1.) and the expected present value of future contributions PV[C(t)] (item a.2.) against the expected present value of future benefit expenditure PV[E(t)] (item b).

It should be noted that there are several methods of evaluating liabilities in respect of the coverage of periods . In our analysis, two methods are used. One is the accrued-to-date liability – alternatively called the "scheme termination" liability – which assumes that the scheme pays only pro rata pensions corresponding to the past contribution credits. (In the actuarial balance sheet, this liability is compared with the reserves since the future contributions are not considered.) The other is the open fund liability – alternatively, the "going-concern" liability – which takes into account not only past service liabilities but also the benefits corresponding to future contribution credits by the currently active workers and the new entrants to the scheme.

Second, the liquidity constraint means that at each period of time the scheme has to ensure the cash income and liquid asset adequate to cover the next payment due.

$$C(t) + H(t) \ge B(t) \qquad (for \ each \ t). \tag{5.4.}$$

Here, H(t) denotes the other cash income. Projections of the fund operation based on the estimated expenditure and tax base are generally used in order to ascertain this condition. The liquidity constraint can usually be expressed in terms of the performance indicators.

For the purpose of financial planning, we are interested in knowing the minimum contribution rate that attains predetermined target values of performance indicators in a given target year. By solving equation (4.5) with the given boundary conditions, we obtain the following formulae.

5.2. Methods of setting long-term contribution rates

In a narrow sence, financing method means methods of setting long-term contribution rates. There are two typical methods.

A (pure) pay-as-you-go method is to set the contribution rate equal to the benefit expenditure in the current year.

$$p(t) = C^{PAYG}(t) = \frac{B(t)}{S(t)} \qquad (for \ each \ t).$$

$$(5.5.)$$

If the fund retains certain reserves then the interest income on the reserves can be used to cover the benefit expenditure, thusby requireing a lower contribution rate. Note that this financing method satisfies both the solvency and liquidity conditions. (Observe that this contribution rate can be derived from the liquidity condition with Q(t) = 0.)

An actuarial level premium is calculated by

$$p(t) = C^{L} = \frac{PV[B(t)] - F_{0}}{PV[S(t)]}.$$
(5.6.)

This contribution rate is derived from the solvency condition. Therefore it satisfies the solvency condition but it does not necessarily meet the liquidity condition. Thus, if this method is applied, the financial manager of a pension fund should ensure that the fund keeps sufficient liquid income.

For the comparison of these methods, it is instructive to draw graghs indicating the time series development of contribution rates p(t) and the reserves F(t). Also it is worthwhile to discuss the financial stability under these two methods in the contect of an increasing trend of pay-as-you-go cost rate (which is plausible under the population ageing)

Between these two extreme cases, a number of intermediate methods are actually applied. One such example is a Scaled Premium method (for the detail, see Appendix B.) Case studies in the United States, Germany, Japan and Canada suggest that required conditions in determining the contribution rate are stated in terms of the financial indicators, in particular, the reserve ratio and the balance ratio. In fact, it was found that a large class of contribution rates (including the above three methods) can be obtained by imposing certain conditions in terms of financial indicators. We call them the Generalized Scaled Premia and will investgate in the following section.

5.3. Generalized Scaled Premium

(1) Statement of the problem

Assume that the initial reserve, assumed future interest rates, and estimated figures of future expenditure and total contributory earnings are given. Then the problem is stated as follows: if a period of equilibrium and target values of financial indicators are given, then one must find the level contribution rate such that during the period of equilibrium the resulting financial indicators are sufficient to meet their target values. Clearly, the higher the contribution rate is, the more likely the above condition is met. Therefore, one must find the lowest possible contribution rate.

Let $n \leq t \leq m$ be the period of equilibrium, and a_0 , b_0 , the target values of the reserve ratio and the balance ratio respectively. Let us denote by $GSP_{n,m}(a \geq a_0, b \leq b_0)$ the lowest contribution rate which satisfies the above conditions, and call it the Generalized Scaled Premium with respect to the period of equilibrium $n \leq t \leq m$ and target values a_0 , b_0 . The term "Generalized Scaled Premium" reflects the fact that it provides a generalization of the Scaled Premium, which is the level contribution rate preserving the accumulated reserves during a given period. In fact, if we denote by $SP_{n,m}$ the Scaled Premium for the same period as above, then $SP_{n,m} = GSP_{n,m}(b \leq 1)$.

(2) Reduction of the problem

In this section, we set out the procedure to determine the Generalized Scaled Premium. For simplicity, we assume $1 \le t \le m$ and put $GSP_m(a \ge a_0, b \le b_0)$ for $GSP_{1,m}(a \ge a_0, b \le b_0)$.

In order to calculate $GSP_m(a \ge a_0)$, we define the contribution rate such that the reserve ratio attains its target value in year t = d $(1 \le d \le m)$, i.e. $a_d = a_0$, and denote it by $Q_d(a = a_0)$.

Then it follows that

$$GSP_m(a \ge a_0) = max\{ Q_d(a = a_0) ; 1 \le d \le m\}.$$
(5.7.)

In fact, if the above contribution rate is lower than one of the values of $\{Q_d(a = a_0) ; d = 1, ..., m\}$, say $Q_h(a = a_0)$ $(1 \le h \le m)$, then it follows that $a_h < a_0$, which means that the condition does not hold in year t = h.

Similarly, we define $Q_d(b = b_0)$ by replacing the reserve ratio by the balance ratio in the definition of $Q_d(a = a_0)$. Then, we have

$$GSP_m(b \le b_0) = max\{ Q_d(b = b_0) ; 1 \le d \le m \}.$$
(5.8.)

Finally, we obtain

$$GSP_m(a \ge a_0, b \le b_0) = max\{GSP_m(a \ge a_0), GSP_m(b \le b_0)\}.$$
(5.9.)

The above discussion indicates the following procedure to calculate the Generalized Scaled Premium:

- (a) Calculate $Q_d(a = a_0)$ and $Q_d(b = b_0)$ for each year of the given period of equilibrium.
- (b) Take the maximum value of them. The result is the required contribution rate.

In practice, experience has shown that Q_d is a monotonically increasing function of d if the PAYG cost rate increases substantially. In such cases, the value at the end year of the period of equilibrium, Q_m , gives the maximum value.

(3) Premium formulae

We develop the formulae for $Q_d(a = a_0)$ and $Q_d(b = b_0)$ in the generic case. You may skip this section The next section discusses the general case.

Let us summarise the given data:

- Assumed interest rate: i.
- Period of equilibrium: $1 \le t \le m$. (predetermined)
- Target values of the reserve ratio and the balance ratio: a_0, b_0 . (predetermined)
- Initial reserve: F_0 .
- Estimated values of expenditure and total contributory earnings: $B(t), S(t) \quad (1 \le t \le m).$

Our strategy is as follows. Equation (4.9.), seen as the recursion formula of the sequence $\{F(t)\}$, describes the evolution of the reserve from one year to the next. If the interest rate and future figures

of total contributory earnings, expenditure and contribution rates are given, then the contribution income, the interest income and the total income are determined by equations (4.1.), (4.2.), (4.3.) and (4.8.). Furthermore, starting with the reserve in the initial year, the reserve in the future years can be obtained by applying formula (4.9.) successively. Consequently, the independent variables for the calculation of the future reserve are $S(t), B(t), p(t), F_0$ and *i*. The liquidity constraint can be expressed in terms of financial indicators. By solving equation (4.9.) with the given boundary conditions, we obtain the required results.

The basic equation of the evolution of the reserve (equation (4.9.)) is rewritten as:

$$vF(t) = F(t-1) + v^{\frac{1}{2}}(pS(t) - B(t)), \qquad (5.10.)$$

where, $v = \frac{1}{1+i}$ is the discount rate. As we are looking for a level contribution rate, the contribution rate on the right-hand side of (5.10.) is assumed to be independent of year t.

The solution of this recursion formula is

$$F(t) = v^{-t} \left(p \sum_{k=1}^{t} v^{k-\frac{1}{2}} S(k) - \sum_{k=1}^{t} v^{k-\frac{1}{2}} B(k) + F_0 \right) \qquad (for \quad t = 0, 1, 2, 3, ...)$$
(5.11.)

(i) Formula for $Q_d(a = a_0)$

The condition is $a(d) = a_0$. Thus, from (4.6.),

$$F(d-1) = a_0 B(d)$$
(5.12.)

Substituting (5.11.) into (5.12.), we obtain the following formula:

$$p = Q_d(a = a_0) = \frac{a_0 v^{d-1} B(d) - F_0 + \sum_{k=1}^{d-1} v^{k-\frac{1}{2}} B(k)}{\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} S(k)},$$
(5.13.)
(for $d = 2, 3, 4, ..., m$).

Putting $a_0 = 0$ in (5.13.) yields the formula of the actuarial level premium for the period $1 \le t \le d-1$. Therefore, the concept of Generalized Scaled Premium also includes the actuarial level premium as a special case.

(ii) Formula for $Q_d(b=b_0)$

The condition is $b(d) = b_0$. Thus, from (4.7.),

$$C(d) - B(d) + b_0 I(d) = 0. (5.14.)$$

Substituting (4.2.) and (4.8.) into (5.14.), we obtain

$$pS(d) - B(d) + b_0[(\sqrt{1+i} - 1)(pS(d) - B(d)) + iF(d-1)] = 0.$$
(5.15.)

By combining (5.11.) and (5.15.), we obtain the following formula:

$$p = Q_d(b = b_0) = \frac{(1 + b_0(v^{-\frac{1}{2}} - 1))v^d B(d) + b_0(1 - v)\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} B(k) - F_0}{(1 + b_0(v^{-\frac{1}{2}} - 1))v^d S(d) + b_0(1 - v)\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} S(k)}, \quad (5.16.)$$
(for $d = 1, 2, 3, .., m$).

Putting $b_0 = 1$ in (5.16.) leads to the formula of the Scaled Premium.

 $Exc(^{**})$. Derive the premium formula that guarantees the predetermined liquidation ratio for a specified target period.

(4) Premium formulae under varying interest rates

1°. Now we generalize the results obtained in the previous section with a view to:

- Consider the situation where the assumed interest rate is time dependent (i.e. i = i(t)).
- Give the formula over an arbitrary period of equilibrium $n \le t \le m$ (n < m).

- Rewrite the formula in a form more suitable for numerical calculation.

 2° . When the assumed interest rate depends on time, the basic equation (equation (5) or (14) in the paper) is written

$$F(t) = (1+i(t))F(t-1) + \sqrt{1+i(t)} \cdot (pS(t) - B(t)), \quad or$$

$$v(t)F(t) = F(t-1) + v(t)^{\frac{1}{2}} \cdot (pS(t) - B(t)), \quad (5.17.)$$

where $v(t) = \frac{1}{1+i(t)}$.

To solve this equation we introduce the following notations:

$$V(t) = \prod_{k=1}^{t} v(k) \quad ; \quad W(t) = V(t-1) \cdot v(t)^{\frac{1}{2}} \qquad (for \quad t = 1, 2, 3, ...).$$
(5.18.)

Put, as a convention, that V(0) = 1.

By multiplying V(t-1) to both sides of the second equation, we have

$$V(t)F(t) = V(t-1)F(t-1) + pW(t)S(t) - W(t)B(t).$$
 (Note $V(t) = V(t-1) \cdot v(t)$). (5.19.)

From this, for $t \geq n$,

$$\sum_{k=n}^{t} [V(k)F(k) - V(k-1)F(k-1)] = \sum_{k=n}^{t} [pW(k)S(k) - W(k)B(k)].$$
(5.20.)

Therefore, we have the solution in the following form:

$$V(t)F(t) = V(n-1)F(n-1) + p(\overline{S(t)} - \overline{S(n-1)}) - (\overline{B(t)} - \overline{B(n-1)})$$
(5.21.)

Here, we put

$$\overline{S(t)} = \sum_{k=1}^{t} S(k)W(k) \quad ; \quad \overline{B(t)} = \sum_{k=1}^{t} B(k)W(k).$$
(5.22.)

3°. We give the formulae for $Q_d(a = a_0)$ and $Q_d(b = b_0)$ over the period of equilibrium $n \le t \le m$. We denote these rates by $Q_d(a = a_0; n, m)$ and $Q_d(b = b_0; n, m)$, respectively.

The given data are summarised as follows:

- Interest rate: i = i(t). (equivalently, discount rate $v(t) = \frac{1}{1+i(t)}$.)
- Period of equilibrium: $n \leq t \leq m$.
- Target values of the reserve ratio and the balance ratio: a_0 and b_0 .
- Initial reserve: F(n-1).
- Projected expenditure and total contributory earnings: B(t) and S(t).

By similar calculation, corresponding to formula (5.13.) we have

$$p = Q_d(a = a_0; n, m) = \frac{a_0 V(d-1)B(d) - V(n-1)F(n-1) + (\overline{B(d-1)} - \overline{B(n-1)})}{\overline{S(d-1)} - \overline{S(n-1)}},$$
(5.23.)

 $(for \quad d = n + 1, n + 2, .., m)$

Corresponding to formula (5.16.), we have

$$p = Q_d(b = b_0; n, m)$$

=
$$\frac{(1 + b_0(v(d)^{-\frac{1}{2}} - 1))V(d)B(d) + b_0(1 - v(d))(\overline{B(d - 1)} - \overline{B(n - 1)}) - V(n - 1)F(n - 1)}{(1 + b_0(v(d)^{-\frac{1}{2}} - 1))V(d)S(d) + b_0(1 - v(d))(\overline{S(d - 1)} - \overline{S(n - 1)})}$$
(5.24.)

 $(for \quad d=n,n+1,...,m)$

4°. We derive a more general formula for $Q_d(b=b_0)$.

Recall the short-term liquidity condition:

$$C(t) + H(t) \ge E(t),$$

where

C(t) : Contribution income (in cash) in year tH(t) : Cash income other than contributions in year tE(t) : Expenditure in year t.

A theoretical range of H(t) is given by:

$$0 \leq H(t) \leq F(t-1) + I(t),$$

where

F(t-1) : Reserve at the beginning of year t I(t) : Investment income in year t. Thus,

$$H(t) = \alpha I(t) + \beta F(t-1),$$

with $\alpha, \beta \in [0, 1]$.

Here, α and β represent the portion of liquidable investment income and that of reserves, respectively.

We define the *liquid ratio* by

$$l(t) = \frac{E(t) - C(t)}{H(t)} = \frac{E(t) - C(t)}{\alpha I(t) + \beta F(t-1)}.$$

In the case where the expenditure exceeds the contributions income, the liquid ratio represents the excess expenditure as a percentage of the total liquidable income. Note that l(t) = b(t) if $\alpha = 1$ and $\beta = 0$.

Consider the GSP that guarantees a given target value of l_0 for a period [n, m] (for given α and β). Then by similar process, we have

$$GSP_{n,m}(l \ge l_0) = \max\{Q_d(l = l_0) ; n \le d \le m\}$$

and

$$Q_d(l=l_0;n,m) = \frac{[1+l_0\alpha(v_d^{-\frac{1}{2}}-1)]V_dE_d + l_0[\alpha(1-v_d)+\beta v_d](\overline{E}_{d-1}-\overline{E}_{n-1}-V_{n-1}F_{n-1})}{[1+l_0\alpha(v_d^{-\frac{1}{2}}-1)]V_dS_d + l_0[\alpha(1-v_d)+\beta v_d](\overline{S}_{d-1}-\overline{S}_{n-1})}$$

(for $d = n, \ldots, m$).

Note that if $\alpha = 1$ and $\beta = 0$, we have $Q_d(l = l_0) = Q_d(b = l_0)$.

(5) Policy for setting the target levels

Mathematically, the Generalized Scaled Premium is obtained by putting additional conditions to the basic equation of the evolution of the reserve. It is thus important to discuss what is the adequate required levels of the reserve ratio and the balance ratio. As shown earlier, each country adopts different requirements and develops its own rules for their national social security pension scheme.

5.4. Discussions on further issues [Note. This section is still a sketchy draft]

We consider the process of actuarial valuation and raise key issues for discussion.

(1) We have assumed that the projected values of future benefit expenditure and contributory salaries are there. But the first task of actuaries is to produce these projected values.

Actuary's problem (Stage I) : Estimate B(t) and S(t) for future time t.

We have simulation models to do this (ILO-POP, ILO-PENS). An issue is how to make demographic and economic assumptions, justify their plausibility and ensure their consistency. (A steady-state analysis)

(2) The next task of actuaries is to determine the long-term contribution rates based on the projected stream.

Actuary's problem (Stage II): Determine p(t) for future time t.

We have seen major financing methods. The issue here is how to choose the suitable financing system, including setting key financial policy variables.

(3) Once the future contribution rates are set out, the task of the financial manager is to maximize the investment return while meeting liquidity requirement. This is a dual problem to the actuary's and can be stated as follows:

Financial manager's problem: Given p(t), find the optimal portfolio in order to (a) maximize the period until the time of exhaustion of the reserve or (b) maximize the resulting reserve for a given period [0, n]

Issue: Pension fund usually is a large investor and has huge influence on the capital market. Matching of liquid return and liability. Pension ALM.

(4) Financial evaluation of reform

Actuarial projections can be used to evaluate the effects of a pension reform. This is done by comparing key reform indicators of before and after the reform.

Contribution rates: $p^{status \ quo}$ and p^{reform} .

Unfunded pension liability: $UFL^{status\ quo}$ and UFL^{reform}

Issue: Actuaries' involvement in the policy making. (from financial evaluation to welfare evaluation)

(5) Check and control

Generally there is a discrepancy between the projected amounts and the actual performance. An expost analysis should be conducted to check whether this is due to trend or fluctuation. A first task is to look into the the accounting: compare the performance and projection.

(6) Extension of scope for macro-economic analysis

One application is to consider the expenditure on social security as a whole in national system of account

(7) Sensitivity tests and stochastic control

In the above approach the estimated future benefit and contributory base are deterministic. To cope with the uncertainty in the demographic and economic assumptions, a usual solution is to undertake a sensitivity analysis based on alternative assumptions.

A better theoretical framework is to consider B(t) and S(t) as random variables Then, we have to treat the solvency and liquidity conditions as a stochastic control problem. For example, the condition of the long-term solvency should now read as.

$$EPV[C(t)] + F_0 = EPV[B(t)].$$

Here $EPV[\cdot]$ is the operation taking the expected present value.

Exc. Does this condition imply the following ?

$$\lim_{t \to \infty} EPV[F(t)] = 0.$$

Exc. Formulate the above problem as that of stochastic optimal control.

Exc. Discuss the case where interest rate (i.e. discount rate) is time dependent, and the case where it is further correlated with B(t) and S(t).

A restriction to this approach is that we probably have to assume tractable model (e.g. Brownian motion) as it seems quite difficult to estimate arbitrary stochastic processes (although we could assume some correlations between certain variables as economic theory predicts).

(8) Issues in inter-generational transfer

In the above approach, the problem of pension financing is viewed as that of a representative social planner (the financial manager) with an infinite time horizon. Although this approximation is sensible in the aggregate level, it is not sufficient when we look into intergenerational redistribution effects of PAYG pension schemes. The generational accounting is the generally used approach for this type of analysis. Whilst the actuarial balance sheet captures the scheme's net surplus or debt in the aggregate, the generational accounting breaks it down into the net pension transfer of each generation. The sum of the generational accounts over all generations, including future new entrants, is thus equivalent to the actuarial balance on the open fund basis.

(8) Economic effects of social security pensions

In addition to income redistribution, social security pensions are thought to have economic implecations such as (i) retirement decision of a worker (labour force supply approach, early retirement) (ii) lifetime income allocation between savings and consumption. (Empirical evidences and their causality analysis.)